THE JOURNAL

OF THE

INDIAN MATHEMATICAL SOCIETY

Editor:

R. VAIDYANATHASWAMY, M.A., D.Sc.

Joint Editor:

C. N. SRINIVASIENGAR, D.Sc.

Collaborators:

K. ANANDA RAO, M.A.; S. C. DHAR, D.Sc.; H. RAFAEL, D.Sc., S.J.; S. R. U. SAVOOR, M.A., D.Sc.; M. R. SIDDIQI, M.A., Ph.D.; T. SURYANARAYANA, M.A.

NEW SERIES

(Issued Quarterly)

Vol. III

1938-39

PRINTED AT THE MADRAS LAW JOURNAL PRESS, MYLAPORE, MADRAS

Annual Subscription: Rs. 6

CONTENTS.

	Pages
Banerjee, D. P. On the Expansion of a Function in a Series of $q_n(z)$ ..	164—168
Bell, Clifford. Plane curves with Pseudo-Rhamphoid Cusps	193—197
Bell, E. T. Notes on Denumerants ..	41—45
Biswas, B. N. Statistics of Photons and Planck's Radiation Formula	82—86
Brij Mohan. Self-Reciprocal Functions involving Laguerre Polynomials	268—270
———**and Sastry, R. V.** An Inversion Formula ..	46—48
Erdelyi, A. Infinite Integrals involving Whittaker Functions	169—181
Ganapathy Iyer, V. On the Average Radial Increase of a certain class of Integral Functions of Order one and Finite Type	87—95
——— On certain Functional Equations ..	312—315
Hans Raj Gupta. Waring's Theorem for Powers of Primes	136—145
——— Note on Dirichlet's L-Functions ..	198—199
Hari Shanker. On the Expansion of the Parabolic Cylinder Function in a Series of the Product of two Parabolic Cylinder Functions ..	226—230
Kibble, W. F. A Bessel Function in terms of Incomplete Gamma Functions	271—294
Kosambi, D. D The Tensor Analysis of Partial Differential Equations	249—253
Krishnaswami Ayyangar, A. A. On the Semi-invariants of two Variates and their Additive Property ..	1—7
Levi, F. W. On a fundamental Theorem of Geometry.	182—192
——— Pairs of Inverse Moduls	295—306

		PAGES
Minakshisundaram, S.	A Tauberian Theorem on (λ, K)-Process of Summation ..	127—130
———	On V. Ramaswami's Tauberian Theorem on Oscillation	131—135
———	On the Theory of Non-Linear Partial Differential Equations of the Parabolic Type ..	237—247
Moghe, D. N.	On the Principle of Conservation of Energy in the Relativistic Theory of Gravitation	231—236
Narasinga Rao, A.	Studies in Turbine Geometry I ..	96—108
Pillai, S. S.	On Waring's Problem VIII ..	205—220
———	On Waring's Problem IX	221—225
———	A note on the paper of Sambasiva Rao ..	266—267
Rajagopal, C. T.	Convergence Theorems for Series of Positive Terms	118—125
Ram Behari.	Infinitesimal Deformation of Ruled Surfaces	68—72
———	Comparison of Sannia's Theory of Line Congruences with Gauss's Theory of Surfaces ..	109—113
———	A note on Laguerre's Function ..	202—204
Ramamurthi, B.	On Linear Complexes of Minimum Rank containing all the Tangents of a Rational Norm Curve	34—40
———	**and Srinivasiengar, C. N.** The Lines of Striction on a Quadric	19—24
Ramanath Mohanty.	The Summation of the Integral conjugate to the Fourier Integral of the Finite Type and of the conjugate Series ..	319—329
Rangaswami, K.	The Theory of the General Contact Circles of a Triangle	254—261
Reissner, E.	Remark on the Theory of Bending of plates of Variable Thickness III ..	200—201
Sambasiva Rao, K.	On a Particular Representation of Integers as Sums of kth Powers ..	262—265
Shabde, N. G.	On some Results involving k_n-Functions	146—151
———	On some Series and Integrals involving k_n Functions	307—311
Shah, S. M.	An Inequality for the Arithmetical Function $g(x)$	316—318

PAGES

Shastri, N. A. On Some Results involving Bateman's Function .. .. .. 8—18

——— Integral Relations between the k-Function with Non-Integral Index and Confluent Hypergeometric Functions .. .. 152—154

——— Relations between some confluent Hypergeometric Functions .. .. 155—163

R. V. Shastry and Brij Mohan. An Inversion Formula .. .. .. 46—48

Srinivasiengar, C. N. On the Zeros of Weierstrass's Non-Differentiable Function .. 114—117

——— **and Ramamurthi, B.** The Lines of Striction on a Quadric .. .. 19—24

Suryanarayan, K.S. On Three Rational Functions on a Riemann Surface .. .. 56—67

Vaidyanathaswamy, R. and T. Venkatarayudu. Canonical Basis for Ideals in a Polynomial Domain over a commutative Ring with finite basis for Ideals .. .. 49—53

Varma, R. S. Some Infinite Integrals involving Weber's Parabolic Cylinder Functions .. 25—33

——— On Laguerre's Polynomials which are Self-reciprocal in the Hankel Transform .. 54—55

——— Some Infinite Series involving Sonine's Polynomial .. .. .. 330—333

Venkatarayudu, T. The Algebra of the eth Power Residues .. .. .. 73—81

——— **and Vaidyanathaswamy, R.** Canonical Basis for Ideals in a Polynomial Domain over a commutative Ring with finite basis for Ideals .. 49—53

Corrigenda 126

Errata 248

ON THE SEMI-INVARIANTS OF TWO VARIATES AND THEIR ADDITIVE PROPERTY

By A. A. KRISHNASWAMI AYYANGAR,

Maharaja's College, Mysore.

[Received 30 August 1937]

1. In a recent paper*, I expressed certain statistical constants of the series

$$f(x, y) \equiv \Sigma A_{pq} x^p y^q = \prod_1^n \left\{ f_i(x, y) \right\}^{m_i}$$

in terms of those of its factors, where A_{pq} denotes the frequency of the simultaneous occurrences of paired values (p, q) of the variates (x, y). This may be called a 'Distribution-Series' in preference to the name 'Correlation-Series' given in my previous paper*; for it is intended as a frame-work to hold in convenient juxtaposition the characters and their frequency so as to enable us to operate on the terms algebraically.

This paper is the outcome of an attempt to investigate a general set of statistical constants of the above series, which can be expressed as the sum of the corresponding constants of the factors. It is proved that such constants are comprised in the linear combinations of Thiele's semi-invariants and my generalizations of them, associated with the given distribution. The remarkable additive property of semi-invariants thus becomes conspicuous as the very foundation and justification of their existence. This property, of course, is not new†; but I here sketch a more natural approach to semi-invariants than in Thiele's discussion, where they seem to have been introduced rather abruptly and their properties appear to be more or less afterthoughts.

2. The problem before us is to express

$$v_{r,s} \equiv \left\{ \left(x\frac{\partial}{\partial x} \right)^r \left(y\frac{\partial}{\partial y} \right)^s \log f \right\}_{x=y=1}$$

and

$$v'_{r,s} \equiv \left\{ \frac{\partial^{r+s} \log f}{\partial x^r \partial y^s} \right\}_{x=y=1}$$

* 'Correlation by the method of factors', *Proc. Ind. Acad. of Sc.* VI A, 1.

† '*Theory of Observations*' by T. N. Thiele, London, 1903. Reprinted in '*Annals of Mathematical Statistics*', 2 (1931), 165-306.

in terms of the product-moments of the distribution. These functions are chosen because they are the sums of the corresponding functions of the factors of 'f'. In other words, they possess the 'additive property' like the logarithmic function.

Let $M_{r,s} \equiv \Sigma A_{pq} p^r q^s$ denote a product-moment of the distribution about (0, 0).

$$M_{r,s} = \left[\left(x\frac{\partial}{\partial x} \right)^r \left(y\frac{\partial}{\partial y} \right)^s f \right]_{x=y=1}$$

$$= \left[D_1^r D_2^s \phi \right]_{t_1=t_2=0},$$

where $x=e^{t_1}$, $y=e^{t_2}$ and $f(x, y)=\phi(t_1, t_2)$, D_1, D_2 denoting partial differentiations with respect to t_1, t_2 respectively. $v_{r,s}$ is the corresponding function associated with $\log \phi$.

But
$$\log \phi = \sum \frac{t_1^r t_2^s}{r!\, s!} \left\{ \frac{\partial^{r+s} \log \phi}{\partial t_1^r \partial t_2^s} \right\}_{t_1=t_2=0}$$

$$= \sum v_{r,s} \frac{t_1^r t_2^s}{r!\, s!},$$

and
$$\phi = \sum M_{r,s} \frac{t_1^r t_2^s}{r!\, s!}.$$

Hence

$$\sum v_{r,s} \frac{t_1^r t_2^s}{r!\, s!} = \log \sum M_{r,s} \frac{t_1^r t_2^s}{r!\, s!}; \qquad (1)$$

$$e^{\Sigma v_{r,s} t_1^r t_2^s / r!\, s!} = \Sigma M_{r,s} t_1^r t_2^s / r!\, s!, \qquad (1')$$

where $v_{00} = \log M_{00}$ and $M_{00} = \Sigma A_{pq}$.

We call the functions $v_{r,s}$ 'Semi-invariants' for two variates. When $t_2=0$, the semi-invariants of Thiele are obtained.

2.1 *To express $v'_{r,s}$ in terms of $v_{r,s}$.*

$$v'_{r,s} = \left\{ \frac{\partial^{r+s} \log f}{\partial x^r \partial y^s} \right\}_{x=y=1}$$

$$= \left\{ (e^{-t_1} D_1)^r (e^{-t_2} D_2)^s \log \phi \right\}_{t_1=t_2=0}.$$

It is easily proved that

$$(e^{-t_i} D_i)^n \log \phi = e^{-n t_i} D_i (D_i - 1)(D_i - 2) \ldots (D_i - n + 1) \log \phi,$$
$$i = 1, 2.$$

Hence

$$v'_{r,s}=\Big\{ e^{-rt_1-st_2}D_1(D_1-1)(D_1-2)\ldots(D_1-r+1)D_2(D_2-1)\ldots$$
$$(D_2-s+1)\log\phi\Big\},\ t_1=t_2=0$$
$$=E_1^{(r)}E_2^{(s)}v_{00},$$

where the operational symbols E_1, E_2 increase by unity the first and second subscripts respectively of v_{00}, and $E_i^{(n)}\equiv E_i(E_i-1)\ldots(E_i-n+1)$, $i=1, 2$. Thus, $v'_{r,s}$ is proved to be a linear combination of the semi-invariants $v_{i,j}$ where $i\leqslant r$, $j\leqslant s$.

For example, $\qquad v'_{11}=v_{11}$; $v'_{12}=v_{12}-v_{11}$;

$$v'_{22}=v_{22}-v_{21}-v_{12}+v_{11}.$$

Since the additive property holds for $v_{r,s}$ and $v'_{r,s}$, it also holds for any linear combinations of $v_{r,s}$ and $v'_{r,s}$.

3. *Formulae connecting $M_{r,s}$ and $v_{r,s}$.* By the usual process of differentiating and equating coefficients of corresponding powers of the variables, we get from (1′),

$$M_{i,j}=\Sigma v_{\alpha,\beta}M_{i-\alpha,j-\beta}\binom{i-1}{\alpha-1}\binom{j}{\beta}, \qquad (2)$$

where the summation extends to all values of $\alpha\leqslant i$, $\beta\leqslant j$ and (α,β), (i,j) are different from $(0,0)$.

E.g. $\qquad M_{40}=v_{40}M_{00}+3v_{30}M_{10}+3v_{20}M_{20}+v_{10}M_{30}. \qquad (3)$

Again, expanding the right side of (1) in powers of t_1, t_2 and comparing coefficients, we get

$$\frac{v_{r,s}}{r!\,s!}=\sum\frac{(-1)^{n-1}(n-1)!}{A^{\pi_1,\pi_2,\ldots}_{i_1i_2\ldots,j_1j_2\ldots}}M^{\pi_1}_{i_1j_1}M^{\pi_2}_{i_2j_2}\ldots/M^n_{00}, \qquad (4)$$

where $\quad A^{\pi_1,\pi_2,\ldots}_{i_1i_2\ldots,j_1j_2\ldots}=(i_1!\,j_1!)^{\pi_1}(i_2!\,j_2!)^{\pi_2}\ldots(\pi_1!\pi_2!\ldots)$

and the summation extends over all positive integral values of i, j, π satisfying simultaneously the equations

$$\Sigma i_1\pi_1=r,\dagger\ \Sigma j_1\pi_1=s,\ \Sigma\pi_1=n$$

the number of terms in each summation being the same, $(i,j)\neq(0,0)$, $\pi\neq0$, and (i,j)'s being all distinct from one another.

Expansion of the left side of (1′) leads to

$$\frac{M_{r,s}}{r!\,s!}\Big/M_{00}=\sum v^{\pi_1}_{i_1j_1}v^{\pi_2}_{i_2j_2}\ldots\Big/A^{\pi_1,\pi_2,\ldots}_{i_1i_2\ldots,j_1j_2\ldots} \qquad (5)$$

† See Appendix for a method of determining the values of i, j, π.

3.1. Comparing (4) and (5), we find that when the M's are expressed in terms of the v's, we can easily derive from this the expressions for the v's in terms of the M's by interchanging the symbols $v_{i,j}$ and $M_{i,j}/M_{00}$ and adjoining to each term in the summation an appropriate coefficient in the form $(-1)^{n-1}(n-1)!$, where n is the degree of the term in the v's. Thus, it is sufficient to tabulate only one set of formulæ, viz. those corresponding to (5) for reading off from it the other set corresponding to (4). This remark is illustrated below:

$v_{r,s}$ *in terms of* $M_{r,s}$	$M_{r,s}$ *in terms of* $v_{r,s}$
$v_{01}=M_{01}/M_{00}$	which reverses into itself.
$v_{11}=\frac{M_{11}}{M_{00}}-\frac{M_{01}M_{10}}{M_{00}^2}$	$\frac{M_{11}}{M_{00}}=v_{11}+v_{01}v_{10}$
$v_{02}=\frac{M_{02}}{M_{00}}-\frac{M_{01}^2}{M_{00}^2}$	$\frac{M_{02}}{M_{00}}=v_{02}+v_{01}^2$
$v_{30}=\frac{M_{30}}{M_{00}}-\frac{3M_{10}M_{20}}{M_{00}^2}+\frac{2M_{10}^3}{M_{00}^3}$	$\frac{M_{30}}{M_{00}}=v_{30}+3v_{10}v_{20}+v_{10}^3$
$v_{21}=\frac{M_{21}}{M_{00}}-\frac{M_{01}M_{20}}{M_{00}^2}-\frac{2M_{10}M_{11}}{M_{00}^2}+\frac{2M_{10}^2M_{01}}{M_{00}^3}$	$\frac{M_{21}}{M_{00}}=v_{21}+v_{01}v_{20}+2v_{10}v_{11}+v_{10}^2v_{01}$
$v_{40}=\frac{M_{40}}{M_{00}}-\frac{3M_{20}^2}{M_{00}^2}-\frac{4M_{30}M_{10}}{M_{00}^2}+\frac{12M_{20}M_{10}^2}{M_{00}^3}-\frac{6M_{10}^4}{M_{00}^4}$	$\frac{M_{40}}{M_{00}}=v_{40}+3v_{20}^2+\ v_{30}v_{10}+6v_{20}v_{10}^2+v_{10}^4$
$v_{31}=\frac{M_{31}}{M_{00}}-\frac{M_{30}M_{01}}{M_{00}^2}-\frac{3M_{10}M_{21}}{M_{00}^2}-\frac{3M_{11}M_{20}}{M_{00}^2}+\frac{6M_{11}M_{10}^2}{M_{00}^3}+\frac{6M_{20}M_{10}M_{01}}{M_{00}^3}-\frac{6M_{10}^3M_{01}}{M_{00}^4}$	$\frac{M_{31}}{M_{00}}=v_{31}+v_{30}v_{01}+3v_{10}v_{21}+3v_{11}v_{20}+3v_{11}v_{10}^2+3v_{20}v_{10}v_{01}+v_{10}^3v_{01}$

$$v_{22}=\frac{M_{22}}{M_{00}}-\frac{2M_{11}^2}{M_{00}^2}-\frac{2M_{01}M_{21}}{M_{00}^2}-\frac{2M_{10}M_{12}}{M_{00}^2}-\frac{M_{02}M_{20}}{M_{00}^2}+\frac{2M_{01}^2M_{20}}{M_{00}^3}+\frac{2M_{10}^2M_{02}}{M_{00}^3}+\frac{8M_{01}M_{10}M_{11}}{M_{00}^3}-\frac{6M_{01}^2M_{10}^2}{M_{00}^4}$$

$$\frac{M_{22}}{M_{00}}=v_{22}+2v_{11}^2+2v_{01}v_{21}+2v_{10}v_{12}+v_{02}v_{20}+v_{01}^2v_{20}+v_{10}^2v_{02}+4v_{01}v_{10}v_{11}+v_{01}^2v_{10}^2$$

N.B. For a check on the coefficients of the v-terms, it may be noticed that the sum of the numerical coefficients is the same for all moments of the same order. In fact, the sum of the numerical coefficients of the various terms in a unit-moment of the n-th order is the coefficient of $x^n/n!$ in the expansion of e^{e^x-1}, to which the left side of (1′) divided out by M_{00} reduces when the v's are made equal to 1 and $t_1+t_2=x$. Again, when all the M's are equal to one another, all the semi-invariants except v_{00}, v_{01} and v_{10} vanish, since the left side of (1′) now reduces to $M_{00}e^{t_1+t_2}$. Hence, the sum of the numerical coefficients in the expressions for the semi-invariants, except v_{00}, v_{01} and v_{10}, in terms of the moments is zero.

4. An examination of the preceding table shows that *the additive property holds, among moments, only for* M_{01}/M_{00} and M_{10}/M_{00}, i.e. *the two means*, whatever the origin may be. For, these are the only functions which are also semi-invariants. But, if the origin be taken at the mean of the distribution, $M_{01}=M_{10}=0$, so that $v_{01}=v_{10}=0$ and *the unit moments of the second and third orders also become equal to the corresponding semi-invariants of the same order and therefore acquire the additive property.*

Advantage may be taken of the additive property for the calculation of product-moments of factorisable distributions such as the Binomial distribution $(P+Qx)^n$, by first calculating the semi invariants for the factors and utilising them for the original distribution. For example, the mean, and the second and third moments about the mean of the binomial distribution are merely n-times the corresponding moments for the simple system of two characters 0, 1 with frequencies P, Q respectively.

APPENDIX

On finding the positive integral solutions of two linear simultaneous equations in several variables.

The simultaneous equations are

$$i_1\pi_1+i_2\pi_2+\ldots+i_k\pi_k=r$$
$$j_1\pi_1+j_2\pi_2+\ldots+j_k\pi_k=s,$$

where r, s alone are given.

The simple type $i\pi=r'$, $j\pi=s'$ has obviously as many solutions as there are factors of the greatest common divisor of and s'.

Consider all the k-partitions of r and s, which may include also zero-elements and associate them one with the other, taking care that the number of zeros in a k-partition of r (or s) does not exceed the number of non-zero elements in the associated k-partition of s (or r). This will secure the ordering of the elements in the two associated partitions so that zero-elements do not occupy the same rank in both. Now solve the k-equations of the type

$$i\pi=r',\ j\pi=s', \tag{1}$$

where r', s' are equal to the elements of the same rank in the associated ordered partitions. Combine these solutions in all possible ways so that no two paired (i, j) values are identical, and we obtain the complete set of solutions, provided all the possible associations of ordered k-partitions for all possible values of k are taken into account. The solutions may be written down compactly in the form

$$(i_1j_1{}^{\pi_1},\ i_2j_2{}^{\pi_2},\ldots)\ ;\ldots.$$

Illustration.

$r=3$, $s=1$. Possible partitions of r and s without zero-elements are r: 3, 21, 111; s: 1.

Ordered k-partitions of (r), (s) including zero-elements may be associated thus:

3	30	21	21	210	111	1110
1,	01,	10,	01,	001,	100	0001.

Distinct solutions of equations of type (1) are as follows:

Equation	*Solution*
$i\pi=3,\ j\pi=1$	31^1
$i\pi=3,\ j\pi=0$	30^1; 10^3
$i\pi=0,\ j\pi=1$	01^1
$i\pi=1,\ j\pi=0$	10^1
$i\pi=2,\ j\pi=1$	21^1
$i\pi=1,\ j\pi=1$	11^1
$i\pi=2,\ j\pi=0$	10^2; 20^1.

The complete set of solutions may now be written down after those that contain repeated values of the pair (i, j) such as $(10^2, 10^1, 01^1)$ are rejected:

$$(31^1);\ (30^1, 01^1);\ (10^3, 01^1);$$
$$(21^1, 10^1);\ (10^2, 11^1);\ (20^1, 11^1);$$

and
$$(20^1, 10^1, 01^1),$$

seven sets in all.

ON SOME RESULTS INVOLVING BATEMAN'S FUNCTION

By N. A. SHASTRI, *College of Science, Nagpur.*

[Received 12 November 1937]

In this note a number of results involving k-functions have been put together. In all cases, methods of operational calculus have been used to obtain the results. It is believed that all the results are new. For theorems of operational calculus used in this note, reference may be made to the papers of the author published in this *Journal* before.

I. We have the operational relation[(1)]

$$(-)^{n-1}2^{-\frac{1}{2}}\Gamma(1+n)\frac{1}{p^{n-1}}e^{-\frac{1}{p}}k_{2n}\left(\frac{1}{p}\right)\doteq x^{n-\frac{1}{2}}J_1(2\sqrt{2x}). \quad (1)$$

Therefore

$$(-)^{n-1}2\Gamma(1+n)\frac{1}{p^{n-1}}e^{-\frac{1}{8p}}k_{2n}\left(\frac{1}{8p}\right)\doteq x^{n-\frac{1}{2}}J_1(\sqrt{x}). \quad (2)$$

But[(2)]

$$x^nJ_m(x)\doteq(-)^m\Gamma(n-m+1)\frac{p}{(1+p^2)^{\frac{n+1}{2}}}P_n^m\left(\frac{p}{\sqrt{1+p^2}}\right),$$

where $n\geqslant m, n$ and m are integers, and P_n^m is an associated Legendre function. Put $m=1$ and change n into $2n-1$; we get

$$x^{2n-1}J_1(x)\doteq-\Gamma(2n-1)\frac{p}{(1+p^2)^n}P_{2n-1}^{(1)}\left[\frac{p}{\sqrt{1+p^2}}\right]. \quad (3)$$

From (2), using a result of the operational calculus[(2)], we get

$$x^{2n-1}J_1(x)\doteq(-)^{n-1}2\Gamma(1+n)\frac{p}{\sqrt{\pi}}\int_0^\infty e^{-s^2\left(\frac{1}{8}+\frac{p^2}{4}\right)}s^{2n-2}k_{2n}\left(\frac{s^2}{8}\right)ds.$$

Hence

$$\frac{(-)^n2\Gamma(1+n)}{\sqrt{\pi}}\int_0^\infty e^{-s^2\left(\frac{1}{8}+\frac{p^2}{4}\right)}s^{2n-2}k_{2n}\left(\frac{s^2}{8}\right)ds$$

$$=\Gamma(2n-1)\frac{1}{(1+p^2)^n}P_{2n-1}^{(1)}\left[\frac{p}{\sqrt{1+p^2}}\right].$$

This result can be transformed into the following by replacing $1+2p^2$ by p and simplifying a little.

$$\int_0^\infty e^{-px}x^{n-\frac{3}{2}}k_{2n}(x)\,dx$$

$$=\frac{(-)^n\Gamma(2n-1)}{\Gamma(1+n)}\frac{\sqrt{\pi}}{2^{2n-\frac{3}{2}}}\frac{1}{(p+1)^n}P^{(1)}_{2n-1}\left[\sqrt{\frac{p-1}{p+1}}\right]. \tag{4}$$

Hence we get a new operational image

$$x^{n-\frac{3}{2}}k_{2n}(x)\doteqdot(-)^n\frac{\Gamma(2n-1)}{\Gamma(n+1)}\frac{\sqrt{\pi}}{2^{2n-\frac{3}{2}}}\frac{p}{(p+1)^n}P^{(1)}_{2n-1}\left[\sqrt{\frac{p-1}{p+1}}\right]. \tag{5}$$

II. We have the result[3]

$$L_n^\alpha(x)L_m^\alpha(y)$$

$$=\frac{\Gamma(n+\alpha+1)}{m!}\sum_{l=0}^{n}\sum_{r=0}^{l}(-)^r\binom{l}{r}\frac{(m-n+l-r)!}{l!(n-l)!}\frac{(xy)^{n-l}x^r}{\Gamma(n+\alpha+1-l)}$$

$$\times L^{\alpha+r+2n-2l}_{m-n+l-r}(y). \tag{6}$$

But

$$L_n^{(1)}(x)=(-)^n(n+1)\frac{e^{\frac{x}{2}}}{x}k_{2n+2}\left(\frac{x}{2}\right). \tag{7}$$

Hence

$$e^{\frac{x+y}{2}}k_{2n+2}\left(\frac{x}{2}\right)k_{2m+2}\left(\frac{y}{2}\right)$$

$$=\frac{n!}{(m+1)!}\sum_{l=0}^{n}\sum_{r=0}^{l}(-)^{m+n+r}\binom{l}{r}\frac{(m-n+l-r)!\,x^{n-l+r+1}}{l!(n-l)!\;(n-l+1)!}y^{n-l+1}$$

$$\times L^{2n-2l+r+1}_{m-n+l-r}(y),$$

or

$$e^{\frac{y-x}{2}}x^nk_{2n+2}\left(\frac{x}{2}\right)k_{2m+2}\left(\frac{y}{2}\right)$$

$$=\frac{n!}{(m+1)!}\sum_{l=0}^{n}\sum_{r=0}^{l}(-)^{m+n+r}\binom{l}{r}\frac{(m-n+l-r)!\,e^{-x}x^{2n-l+r+1}}{l!(n-l)!\,(n-l+1)!}$$

$$\times y^{n-l+1}L^{2n-2l+r+1}_{m-n+l-r}(y).$$

Putting $x=\frac{2}{p}$ in the above equation we get

$$2^n\frac{1}{p^n}e^{-\frac{1}{p}}k_{2n+2}\left(\frac{1}{p}\right)e^{\frac{y}{2}}k_{2m+2}\left(\frac{y}{2}\right)$$

$$=\frac{n!}{(m+1)!}\sum_{l=0}^{n}\sum_{r=0}^{l}(-)^{m+n+r}\binom{l}{r}\frac{(m-n+l-r)!\,e^{-\frac{2}{p}}}{l!(n-l)!(n+1-l)!}$$

$$\times\left(\frac{2}{p}\right)^{2n-l+r+1}y^{n-l+1}L_{m-n+l-r}^{2n-2l+r+1}(y). \qquad (8)$$

But

$$\left(\frac{2}{p}\right)^{2n-l+r+1}e^{-\frac{2}{p}}\doteqdot(2x)^{\frac{2n-l+r+1}{2}}J_{2n-l+r+1}(2\sqrt{2x}). \qquad (9)$$

Interpreting both sides of (8) with the help of (9) and (1), we get

$$\frac{(-)^n(2x)^{n+\frac{1}{2}}}{\Gamma(n+2)}J_1(2\sqrt{2x})e^{\frac{y}{2}}k_{2m+2}\left(\frac{y}{2}\right)$$

$$=\frac{n!}{(m+1)!}\sum_{l=0}^{n}\sum_{r=0}^{l}(-)^{m+n+r}\binom{l}{r}\frac{(m-n+l-r)!(2x)^{\frac{2n-l+r+1}{2}}}{l!(n-l)!(n+1-l)!}$$

$$\times J_{2n-l+r+1}(2\sqrt{2x})y^{n-l+1}L_{m-n+l-r}^{2n-2l+r+1}(y),$$

which reduces after a slight simplification to

$$e^{\frac{y}{2}}J_1(2\sqrt{2x})k_{2m+2}\left(\frac{y}{2}\right)$$

$$=\frac{n!(n+1)!}{(m+1)!}\sum_{l=0}^{n}\sum_{r=0}^{l}(-)^{m+r}\binom{l}{r}\frac{(m-n+l-r)!(2x)^{\frac{r-l}{2}}}{l!(n-l)!(n+1-l)!}$$

$$\times J_{2n-l+r+1}(2\sqrt{2x})y^{n-l+1}L_{m-n+l-r}^{2n-2l+r+1}(y). \qquad (10)$$

III. Putting $n=1$ in (1) we get

$$\sqrt{2x}\,J_1(2\sqrt{2x})\doteqdot e^{-\frac{1}{p}}k_2\left(\frac{1}{p}\right).$$

Hence

$$\sqrt{2xt}\,J_1(2\sqrt{2xt})\doteqdot e^{-t/p}k_2\left(\frac{t}{p}\right)$$

$$\sqrt{2xt}\,e^tJ_1(2\sqrt{2xt})\doteqdot e^{t(1-\frac{1}{p})}k_2\left(\frac{t}{p}\right).$$

Let

$$t\doteqdot 1/q.$$

Hence

$$k_2(t) \doteqdot \frac{2q}{(1+q)^2}$$

$$k_2\left(\frac{t}{p}\right) \doteqdot \frac{2pq}{(1+pq)^2}$$

and

$$e^{t(1-\frac{1}{p})}k_2\left(\frac{t}{p}\right) \doteqdot \frac{2}{pq}\left(1+\frac{2-p}{pq}\right)^{-2}$$

$$= \sum_{r=0}^{\infty} (-)^r (r+1) \frac{1}{q^{r+1}} \frac{2(2-p)^r}{p^{r+1}}. \tag{11}$$

But

$$\frac{1}{q^{r+1}} \doteqdot \frac{t^{r+1}}{(r+1)!} \tag{12}$$

$$k_{2n}(x) \doteqdot \frac{2p(1-p)^{n-1}}{(1+p)^{n+1}}$$

and

$$e^x k_{2n}(x) \doteqdot \frac{2(2-p)^{n-1}}{p^n}.$$

Interpreting the equation (11) with the help of these results, we get

$$\sqrt{2xt}\; e^t J_1(2\sqrt{2xt}) = \sum_{r=0}^{\infty} \frac{(-)^r}{r!} t^{r+1} e^x k_{2r+2}(x). \tag{13}$$

This result can also be deduced from equation (15) of Howell's paper by using the relation (7). The method used here is different from Howell's method inasmuch as our method involves the use of operational calculus of two variables.

IV. We have

$$e^{-x}k_{2n}(x) \doteqdot (-)^{n-1} \frac{2p^n}{(p+2)^{n+1}}. \tag{14}$$

Therefore

$$e^{-xy^2}k_{2n}(xy^2) \doteqdot (-)^{n-1} \frac{2y^2p^n}{(p+2y^2)^{n+1}}.$$

Hence

$$(-)^{n-1}\, 2\int_0^1 \frac{P_l(1-2y^2)y^2p^n dy}{(p+2y^2)^{n+1}}$$

$$=\sum_{r=0}^{\infty}(-)^{r+n-1}\frac{(n+r)!}{r!\,n!}\left(\frac{2}{p}\right)^{r+1}\int_0^1 P_l(1-2y^2)y^{2r+2}dy$$

$$=\sum_{r=0}^{\infty}\tfrac{1}{2}\frac{(-)^{l+n+r-1}(n+r)!\,\Gamma^2(r+\frac{3}{2})}{r!\,n!\,\Gamma(r-l+\frac{3}{2})\,\Gamma(r+l+\frac{5}{2})}\left(\frac{2}{p}\right)^{r+1} \tag{15}$$

using the result

$$\int_0^1 P_n(1-2y^2)y^{2m}dy=\frac{(-)^n\,\Gamma^2(m+\frac{1}{2})}{2\Gamma(m-n+\frac{1}{2})\,\Gamma(m+n+\frac{3}{2})}.$$

Interpreting both sides of (15) with the aid of (14) we have

$$\int_0^1 e^{-xy^2}k_{2n}(xy^2)P_l(1-2y^2)dy$$

$$=\sum_{r=0}^{\infty}\frac{(-)^{l+n+r-1}\Gamma(n+r+1)\,\Gamma^2(r+\frac{3}{2})2^r x^{r+1}}{\Gamma(r+1)\,\Gamma(n+1)\,\Gamma(r+2)\,\Gamma(r-l+\frac{3}{2})\,\Gamma(r+l+\frac{5}{2})}. \tag{16}$$

V. From (14) and a theorem[2] of operational calculus we get

$$e^{-x^2}k_{2n}(x^2) \doteqdot \frac{2p(-)^{n-1}}{\sqrt{\pi}}\int_0^{\infty}e^{-\frac{p^2s^2}{4}}\frac{1}{s^{2n}}\left(2+\frac{1}{s^2}\right)^{-n-1}ds$$

$$=2(-)^{n-1}\frac{p}{\sqrt{\pi}}\int_0^{\infty}e^{-\frac{p^2s^2}{4}}s^2(1+2s^2)^{-n-1}ds$$

$$=(-)^{n-1}\frac{8}{p^2\sqrt{\pi}}\int_0^{\infty}e^{-t}t^{\frac{1}{2}}\left(1+\frac{8t}{p^2}\right)^{-n-1}dt$$

$$=(-)^{n-1}\tfrac{1}{2}\left(\frac{p^2}{8}\right)^{\frac{n}{2}-\frac{1}{4}}e^{\frac{p^2}{16}}\,W_{-\frac{3}{4}-\frac{n}{2},\,\frac{1}{4}-\frac{n}{2}}\left(\frac{p^2}{8}\right)$$

using the result [4]

$$W_{k,m}(z)=\frac{e^{-\frac{1}{2}z}z^k}{\Gamma(\frac{1}{2}-k+m)}\int_0^{\infty}e^{-t}\,t^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}dt$$

$$R(k-\tfrac{1}{2}-m)\leqslant 0. \tag{17}$$

Hence

$$e^{-x^2}k_{2n}(x^2)\doteqdot(-)^{n-1}\tfrac{1}{2}\left(\frac{p^2}{8}\right)^{\frac{n}{2}-\frac{1}{4}}e^{\frac{p^2}{16}}\,W_{-\frac{3}{4}-\frac{n}{2},\,\frac{1}{4}-\frac{n}{2}}\left(\frac{p^2}{8}\right). \tag{18}$$

VI. Using

$$e^{-\frac{x}{2}}k_{2n}\left(\frac{x}{2}\right) \doteq \frac{(-)^{n-1}p^n}{(p+1)^{n+1}} \tag{19}$$

with a theorem[2] of operational calculus, we get

$$\frac{1}{\sqrt{\pi x}}\int_0^\infty e^{-\frac{s^2}{4x}-\frac{s}{2}}k_{2n}\left(\frac{s}{2}\right)ds \doteq \frac{(-)^{n-1}p^{\frac{n}{2}}}{(1+\sqrt{p})^{n+1}}$$

$$= \sum_{r=1}^{n-1}(-)^r\binom{n-1}{r}\frac{\sqrt{p}}{(1+\sqrt{p})^{n-r+}}$$

$$\doteq \sum_{r=0}^{n-1}(-)^r\binom{n-1}{r}\sqrt{\frac{2}{\pi}}\,(2x)^{\frac{1}{2}(n-r)}\,e^{\frac{1}{2}x}D_{-n+r-1}(\sqrt{2x})$$

after interpreting the right side with the help of[5]

$$\frac{\sqrt{p}}{(1+\sqrt{p})^n} \doteq \sqrt{\frac{2}{\pi}}(2x)^{\frac{1}{2}(n-1)}e^{\frac{1}{2}x}D_{-n}(\sqrt{2x}). \tag{20}$$

Therefore

$$\int_0^\infty e^{-\frac{s^2}{4x}-\frac{s}{2}}k_{2n}\left(\frac{s}{2}\right)ds$$

$$= \sum_{r=0}^{n-1}(-)^r\binom{n-1}{r}(2x)^{\frac{1}{2}(n+1-r)}e^{\frac{1}{2}x}D_{-n+r-1}(\sqrt{2x}). \tag{21}$$

A more general result can be obtained as follows:

$$\frac{2}{\sqrt{\pi}(2\sqrt{x})^{m+1}}\int_0^\infty e^{-\frac{s^2}{4x}-\frac{s}{2}}k_{2n}\left(\frac{s}{2}\right)H_{e_m}\left(\frac{s}{2\sqrt{x}}\right)ds^{(6)}$$

$$\doteq \frac{(-)^{n-1}p^{\frac{m+n}{2}}}{(1+\sqrt{p})^{n+1}}$$

$$= \sum_{r=0}^{m+n-1}(-)^{r+m}\binom{m+n-1}{r}\frac{\sqrt{p}}{(1+\sqrt{p})^{n-r+1}}$$

$$\doteq \sum_{r=0}^{m+n-1}(-)^{r+m}\binom{m+n-1}{r}\sqrt{\frac{2}{\pi}}\,(2x)^{\frac{1}{2}(n-r)}e^{\frac{1}{2}x}D_{-n+r-1}(\sqrt{2x}).$$

Hence

$$\int_0^\infty e^{-\frac{s^2}{4x}-\frac{s}{2}} k_{2n}(s/2) H_{e_m}\left(\frac{s}{2\sqrt{x}}\right) ds$$

$$= \sum_{r=0}^{m+n-1} (-)^{r+m} \binom{m+n-1}{r} 2^{\frac{m}{2}} (2x)^{\frac{1}{2}(m+n-r+1)} e^{\frac{1}{2}x} D_{-n+r-1}(\sqrt{2x}), \tag{22}$$

where H_{e_m} is the Hermite polynomial.

VII. Applying (7) to

$$L_n^\alpha(x) L_m^\alpha(x)$$

$$= \frac{\Gamma(n+\alpha+1)\,\Gamma(m+\alpha+1)}{\Gamma(n+m+\alpha+1)} \sum_{l=0}^{n \text{ or } m} \frac{(n+m-2l)!\, x^{2l} L_{n+m-2l}^{\alpha+2l}(x)}{l!\,(n-l)!\,(m-l)!\,\Gamma(l+\alpha+1)}$$

we get

$$(-)^{n+m} \frac{1}{x} k_{2n+2}\left(\frac{x}{2}\right) k_{2m+2}\left(\frac{x}{2}\right)$$

$$= \frac{n!\,m!}{(n+m+1)!} \sum_{l=0}^{n \text{ or } m} \frac{(n+m-2l)!\, e^{-x} x^{2l+1} L_{n+m-2l}^{2l+1}(x)}{l!\,(n-l)!\,(m-l)!\,(l+1)!}. \tag{23}$$

But[7]

$$e^{-x} x^{2l+1} L_{n+m-2l}^{2l+1}(x) \doteqdot \frac{(n+m+1)!}{(n+m-2l)!} \frac{p^{n+m-2l+1}}{(p+1)^{n+m+2}}. \tag{24}$$

Applying (24) to (23) we have

$$(-)^{n+m} \frac{1}{x} k_{2n+2}\left(\frac{x}{2}\right) k_{2m+2}\left(\frac{x}{2}\right)$$

$$\doteqdot n!\,m! \sum_{l=0}^{n \text{ or } m} \frac{1}{l!\,(n-l)!\,(m-l)!\,(l+1)!} \frac{1}{p} \cdot \frac{1}{p^{2l-1}} \frac{p^{n+m+1}}{(p+1)^{n+m+2}}.$$

We now interpret the right side with the aid of (19), (11), and the product theorem of the operational calculus, and get

$$\frac{1}{x} k_{2n+2}\left(\frac{x}{2}\right) k_{2m+2}\left(\frac{x}{2}\right)$$

$$= n!\,m! \sum_{l=0}^{n \text{ or } m} \frac{1}{l!\,(l+1)!\,(2l-1)!\,(n-l)!\,(m-l)!}$$

$$\times \int_0^x (x-\xi)^{2l-1} e^{-\xi/2} k_{2n+2m+2}(\xi/2)\, d\xi;$$

or changing x into $2x$ and ξ into 2ξ we get

$$\frac{1}{x}k_{2n+2}(x)k_{2m+2}(x)$$

$$=\sum_{l=0}^{n \text{ or } m}\frac{n!\,m!\,2^{2l+1}}{l!\,(l+1)!\,(2l-1)!\,(n-l)!\,(m-l)!}$$

$$\times\int_0^x(x-\xi)^{2l-1}e^{-\xi}k_{2n+2m+2}(\xi)d\xi. \qquad (25)$$

VIII. Substituting for $x^{\alpha+2l}L_{n+m+\alpha+1}^{\alpha+2l}(x)$ the operational image

$$\frac{\Gamma(n+m+2\alpha+2l+2)}{\Gamma(n+m+\alpha+2)}\frac{(p-1)^{n+m+\alpha+1}}{p^{n+m+2\alpha+2l+1}}$$

on the right side of the equation

$$n!\,m!\,e^{-x}x^{\alpha}L_n^{\alpha}(x)L_m^{\alpha}(x)$$

$$=\sum_{l=0}^{\infty}\frac{\Gamma(n+\alpha+l+1)\,\Gamma(m+\alpha+l+1)\,\Gamma(n+m+\alpha+2)}{l!\,\Gamma(l+\alpha+1)\,\Gamma(n+m+2\alpha+2l+2)}x^{\alpha+2l}$$

$$\times L_{n+m+\alpha+1}^{\alpha+2l}(x)$$

we get

$$n!\,m!\,e^{-x}x^{\alpha}L_n^{\alpha}(x)L_m^{\alpha}(x)$$

$$\doteqdot\sum_{l=0}^{\infty}\frac{\Gamma(n+\alpha+l+1)\,\Gamma(m+\alpha+l+1)}{l!\,\Gamma(l+\alpha+1)}\frac{(p-1)^{n+m+\alpha+1}}{p^{n+m+2\alpha+2l+1}}.$$

Therefore

$$n!\,m!\,e^{-2x}x^{\alpha}L_n^{\alpha}(x)L_m^{\alpha}(x)$$

$$\doteqdot\sum_{l=0}^{\infty}\frac{\Gamma(n+\alpha+l+1)\,\Gamma(m+\alpha+l+1)}{l!\,\Gamma(l+\alpha+1)}\frac{p^{n+m+\alpha+2}}{(p+1)^{n+m+2\alpha+2l+2}} \qquad (26)$$

and

$$n!\,m!\,e^{-2ax}(ax)^{\alpha}L_n^{\alpha}(ax)L_m^{\alpha}(ax)$$

$$\doteqdot\sum_{l=0}^{\infty}\frac{\Gamma(n+\alpha+l+1)\,\Gamma(m+\alpha+l+1)}{l!\,\Gamma(l+\alpha+1)}\frac{a^{-n-m-\alpha-2}\,p^{n+m+\alpha+2}}{\left(1+\frac{p}{a}\right)^{n+m+2\alpha+2l+2}}.$$

But

$$e^{-x}\doteqdot\frac{p}{p+1}.$$

Hence by using Parseval's theorem that if

$$f_1(x) \doteqdot \phi_1(p),\ f_2(x) \doteqdot \phi_2(p)$$

then

$$\int_0^\infty \frac{f_1\phi_2}{x}\,dx = \int_0^\infty \frac{f_2\phi_1}{x}\,dx,$$

we get

$$n!\,m!\int_0^\infty e^{-2ax}(ax)^\alpha L_n^\alpha(ax)\,L_m^\alpha(ax)\frac{dx}{1+x}$$

$$=\sum_{l=0}^\infty \frac{\Gamma(n+\alpha+l+1)\,\Gamma(m+\alpha+l+1)}{l!\,\Gamma(\alpha+l+1)}\,a^{-n-m-\alpha-2}$$

$$\times\int_0^\infty e^{-x}x^{n+m+\alpha+1}\left(1+\frac{x}{a}\right)^{-n-m-2\alpha-2l-2}dx$$

$$=\sum_{l=0}^\infty \frac{\Gamma(n+\alpha+l+1)\,\Gamma(m+\alpha+l+1)\,\Gamma(n+m+\alpha+2)}{l!\,\Gamma(\alpha+l+1)}e^{\frac{1}{2}a}\,a^{l+\frac{\alpha}{2}-\frac{1}{2}}$$

$$\times W_{-l-n-m-\frac{3\alpha}{2}-\frac{8}{2},\,-\frac{1}{2}(\alpha+2l)} \qquad (27)$$

using (17).

We can transform (27) into the following equation by using (7).

$$(-)^{n+m}\frac{(n+1)!\,(m+1)!}{(n+m+2)!}\int_0^\infty \frac{e^{-ax}k_{2n+2}\left(\frac{ax}{2}\right)k_{2m+2}\left(\frac{ax}{2}\right)}{x(1+x)}\,dx$$

$$=\sum_{l=0}^\infty \frac{(n+l+1)!\,(m+l+1)!}{l!\,(l+1)!}e^{\frac{1}{2}a}a^{l+1}W_{-n-m-l-3,\,-\frac{1}{2}(2l+1)}(a). \qquad (28)$$

IX. If we apply Parseval's theorem to (26) and [8]

$$2^{s/2}\Gamma\left(\frac{s+1}{2}\right)p^{\frac{1}{2}}e^{\frac{1}{2}p}D_{-s}(\sqrt{2p}) \doteqdot \frac{x^{\frac{1}{2}(s-1)}}{(1+x)^{s/2}} \qquad (29)$$

we get

$$n!\,m!\,2^{s/2}\,\Gamma\left(\frac{s+1}{2}\right)\int_0^\infty e^{-\frac{3x}{2}}L_n^\alpha(x)L_m^\alpha(x)x^{\alpha-\frac{1}{2}}D_{-s}(\sqrt{2x})\,dx$$

$$=\sum_{l=0}^\infty \frac{\Gamma(n+\alpha+l+1)\,\Gamma(m+\alpha+l+1)}{l!\,\Gamma(l+\alpha+1)}\int_0^\infty \frac{x^{n+m+\alpha+\frac{s}{2}+\frac{1}{2}}}{(1+x)^{n+m+2\alpha+2l+2+\frac{s}{2}}}\,dx$$

$$=\sum_{l=0}^\infty\left[\frac{\Gamma(n+\alpha+l+1)\,\Gamma(m+\alpha+l+1)}{l!\,\Gamma(l+\alpha+1)}\right.$$

$$\left.\times\frac{\Gamma(n+m+\alpha+\frac{s}{2}+\frac{8}{2})\,\Gamma(\alpha+2l+\frac{1}{2})}{\Gamma(n+m+2\alpha+2l+\frac{s}{2}+2)}\right]. \qquad (30)$$

Transforming (30) with the help of (7), we have

$$(-)^{n+m}(n+1)!\,(m+1)!\,2^{s/2}\Gamma\left(\frac{s+1}{2}\right)$$

$$\times\int_0^\infty \frac{e^{-\frac{x}{2}}}{x^{3/2}} k_{2n+2}\left(\frac{x}{2}\right) k_{2m+2}\left(\frac{x}{2}\right) D_{-s}(\sqrt{2x})\,dx$$

$$=\sum_{l=0}^{\infty} \frac{\Gamma(n+l+2)\,\Gamma(m+l+2)\,\Gamma(n+m+\frac{s}{2}+\frac{5}{2})\,\Gamma(2l+\frac{8}{2})}{l!\,\Gamma(l+2)\,\Gamma(n+m+2l+\frac{s}{2}+4)}. \quad (31)$$

X. The following miscellaneous results involving the k-functions can be derived immediately from the results obtained by Howell by using (7).

$$e^x k_{2n+2}(x) = \frac{(-)^n}{n+1}\sum_{r=0}^{n} \frac{\Gamma(r+1-\beta)}{\Gamma(1-\beta)r!} 2xL_{n-r}^{\beta}(2x). \quad (32)$$

$$e^x k_{2n+2}(x) = \frac{2}{n+1}\sum_{r=0}^{n} \frac{(-)^{n+r}}{r!} x^{r+1} L_{n-r}^{r+1}(x). \quad (33)$$

$$e^x k_{4n+2}(x) = \sum_{r=0}^{n} \frac{\Gamma(r+1)}{\Gamma(2r+2)} (2x)^{2r+1} L_{n-r}^{2r}(2x). \quad (34)$$

$$\int_0^\infty e^{-x} x^{s-1} k_{2n+2}(x)\,dx = \frac{(-)^n}{2^s(n+1)!} \frac{\Gamma(n-s+1)\,\Gamma(1+s)}{\Gamma(1-s)}. \quad (35)$$

$$\frac{(-)^{n+m} e^{\frac{(a+b)x}{2}}}{abx} k_{2n+2}(ax/2)\, k_{2m+2}(bx/2)$$

$$\doteqdot \frac{(n+m+1)!}{(n+1)!\,(m+1)!} \frac{(p-a)^n (p-b)^m}{p^{n+m+1}}$$

$$\times F\left(-n,\ -m\,;\ -n-m-1\,; \frac{p(p-a-b)}{(p-a)(p-b)}\right). \quad (36)$$

$$\frac{1}{x} k_{2n+2}(x/2)\, k_{2m+2}(x/2)$$

$$\doteqdot \frac{(-)^{n+m}(n+m+1)!}{(n+1)!\,(m+1)!} \frac{p^{n+m+1}}{(1+p)^{n+m+2}}$$

$$\times F(-n,\ -m\,;\ -n-m-1\,;\, 1-1/p^2). \quad (37)$$

$$\frac{1}{x} k_{2n+2}(x/2)\, k_{2m+2}(x/2)$$

$$\doteqdot \frac{(-)^{n+m}(p-1)^{n+m+2}}{p^{n+m+3}} F\left[n+2,\ m+2\,;\ 2\,;\ \frac{1}{p^2}\right]. \quad (38)$$

$$\frac{1}{x}k_{2n+2}\left(\frac{x}{2}\right)k_{2m+2}\left(\frac{x}{2}\right)$$

$$\doteqdot \frac{(-)^{n+m}p^{n+m+1}}{(1+p)^{n+m+2}} F\left[-n, -m ; 2 ; \frac{1}{p^2}\right]. \qquad (39)$$

REFERENCES.

(1) Shastri.—*Phil. Mag.* (7) XX (1935), 477.
(2) P. Humbert.—*Le Calcul Symbolique*, 27-31.
(3) Howell.—*Phil. Mag.* (7) XXIV (1937).
(4) Whittaker and Watson.—*Modern Analysis*, Chap. XVI, 340.
(5) Varma.—*Jour. L. M. S.* XII. (1937), 26.
(6) Niessen.—*Phil. Mag.* (7) XX(1935), 980.
(7) Shastri.—*Jour. I. M. S.* I (1937), (New series), 236.
(8) Varma.—*Phil. Mag.* (7) XXII(1936), 33.

THE LINES OF STRICTION ON A QUADRIC

By B. RAMAMURTI and C. N. SRINIVASIENGAR.

[Received 3 November 1937]

1. It is well known that the line of striction of a regulus on a quadric is a rational quartic of type (1, 3) cutting every generator of the regulus at a single point and every generator of the other regulus in three points. The rational quartic curve R^4 in [3] has been studied in detail from the projective point of view.* In this paper, the metrical specialisations consequent to the metrical definition of the line of striction, and a few allied problems are considered.

We shall first briefly set down some of the main facts in the projective geometry of R^4†. Any R^4 is given by

$$x_i = f_i(t),\ i = 1, 2, 3, 4,$$

where the f's are quartic polynomials in t. Let $\phi(t)$ be the unique quartic which is apolar to all the f's. Any plane section of R^4 corresponds to a quartic apolar to ϕ and conversely. In particular, the roots of $\phi(t)=0$ correspond to the *flex-points* of R^4, i.e. points where the osculating planes have contact of the third order with the curve. This follows from the fact that if $t=\alpha$ be any root of the equation $\phi(t)=0$, then $\phi(t)$ is apolar to $(t-\alpha)^4$. The projective geometry of R^4 is thus equivalent to the binary geometry of $\phi(t)$. $\phi(t)$ is carried over into itself by the syzygetic group, consisting of Identity, and the three mutually apolar involutions corresponding to the three pairs of quadratics into which ϕ can be split. The double points of the three involutions p_x^2, q_x^2, r_x^2 form the sextic covariant of ϕ. The chords p_x^2, q_x^2, r_x^2 of R^4 are concurrent at a point O, and their polar lines with respect to the quadric containing R^4 lie on the polar plane of O. In other words, the chords and their polar lines form

* Telling, *The Rational Quartic Curve in Space of Three and Four Dimensions*, Cambridge Tract, (No. 34).

† An R^4 may also be obtained as the curve of intersection of two quadrics which touch at a point. Such R^4's do not come up in the topics discussed in this paper.

pairs of opposite sides of a tetrahedron which is called the *Principal Tetrahedron* of R^4. The group of binary collineations carrying R^4 into itself is induced on R^4 by the group of collineations formed by identity and the three harmonic inversions for each of which a pair of opposite edges of the principal tetrahedron are lines of fixed points.

2. Let us now obtain the group of a line of striction L^4 on a real hyperboloid of one sheet Q, purely from metrical considerations. The metrical transformations carrying Q into itself form a group of order eight consisting of Identity, the reflexion P_0 about the centre, the reflexions P_x, P_y, P_z about the axes, and the reflexions Q_x, Q_y, Q_z about the principal planes. Of these the subgroup $1P_xP_yP_z$ carries each regulus into itself and hence each of the two lines of striction into itself. If the principal axes OX, OY, OZ meet the plane at infinity in X, Y, Z respectively, P_x may be regarded projectively as the harmonic inversion for which the opposite edges OX and YZ are lines of fixed points. Hence *the principal tetrahedron of the line of striction is identical with the principal tetrahedron of the quadric.*

Therefore, if $P(x, y, z)$ be a point on L, referred to the principal axes of Q, then $P_1(x, -y, -z)$, $P_2(-x, y, -z)$, $P_3(-x, -y, z)$ are also points on L, and $PP_1P_2P_3$ form a tetrahedron whose axes (*i e.* the lines joining middle points of opposite edges) are the axes of reference. If t be the parameter of P on L, the parameters of P_1, P_2, P_3 may be taken as $t^{-1}, -t, -t^{-1}$ respectively. By varying the point P, we get a syzygetic pencil of tetrahedra given parametrically by $t^4+2\alpha t^2+1=0$. The extremities of the principal axes of the quadric correspond to the parametric values ± 1, $(0, \infty)$, and $\pm\sqrt{-1}$.

These facts are verified by writing down the equations of one of the lines of striction on the hyperboloid of one sheet $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ in the form*

$$\frac{x}{a}:\frac{y}{b}:\frac{z}{c}:w=(1+A)(\lambda^3+\lambda):\lambda^4-1:(1-A)(\lambda^3-\lambda):\lambda^4+2A\lambda^2+1,$$

where $$A=(2b^{-2}+c^{-2}-a^{-2})/(a^{-2}+c^{-2}).$$

* Sommerville, *Analytical Geometry of Three Dimensions*, p. 305.

Let us now trace the curve L. We know* that it passes through the extremities of the three principal axes, and also the points of contact with the conic section q at infinity, of the common tangents of q and the circle at infinity. The latter four points as well as the intersections of L with the z-axis are imaginary. Let us start from the value $\lambda=-\infty$ which corresponds to the vertex $B(0, b, 0)$. As λ varies in the interval $(-\infty, -1)$, the tracing point moves from B to $A'(-a, 0, 0)$. If we suppose that $a>b$, it is easy to see that z is positive or negative according as $A>1$, or $A<1$†. In other words, the part BA' of the curve L^4 lies above or below the principal elliptic section according as $A>1$ or $A<1$. The curve is similarly traced from A' to $B'(0, -b, 0)$, from B' to $A(a, 0, 0)$, and from A to B, as λ changes in the intervals $(-1, 0)$, $(0, 1)$, $(1, \infty)$ respectively. Hence, or by using the fact that the curve is symmetrical about the axes it follows that *the curve is a closed curve which is alternately above and below the principal elliptic section in four quadrants.*

The second line of striction of the quadric has a simple relation with reference to the first. Of the group of metric transformations carrying the quadric into itself, the sub-group $1P_xP_yP_z$ carries each regulus into itself, and hence each line of striction into itself. The remaining transformations P_0, Q_x, Q_y, Q_z convert each regulus into the other, and hence interchange the two lines of striction. It follows that *the two lines of striction of a quadric are reflexions of each other about the centre and about the principal planes of the quadric.* One line of striction lies in four octants, the other line of striction lies in the remaining four octants formed by the co-ordinate axes. From these considerations, we at once get the result which has been otherwise deduced elsewhere, viz. the two lines of striction intersect at the vertices of the quadric, and at the four points at infinity.

3. The equation of the osculating plane to the curve L^4 at the point λ is given by

$$bc(1-A)x\left\{A+3(\lambda^2-\lambda^4)(A+2)-A\lambda^6\right\}$$
$$+ab(1+A)z\left\{A+3(\lambda^2+\lambda^4)(2-A)+A\lambda^6\right\}$$
$$-ac(1-A^2)y(3\lambda^5-2A\lambda^3+3\lambda)-3abc(1-A^2)w(\lambda-\lambda^5)=0.$$

* *Math. Student* 4, 51; *Bull. Calcutta Math. Soc.* 28 (1936), 49.

† The case $A=1$ leads to the surface of revolution, whose either line of striction coincides with the principal elliptic section.

The condition that the binormal at λ should be perpendicular to the normal to the quadric is worked out to be

$$\lambda(1-\lambda^8)\left(\frac{A}{a^2}-\frac{A}{c^2}+\frac{3}{b^2}\right)+\lambda^3(1-\lambda^4)\left(\frac{4A+6}{a^2}+\frac{4A-6}{c^2}-\frac{2A}{b^2}\right)=0.$$

In addition, the point $\lambda=\infty$ also satisfies the condition, for the osculating plane at $(0, b, 0)$ is $x/a(1+A)=z/c(1-A)$ while the normal is the y-axis.

Hence, *there are ten points on the curve at which the principal normal of the curve coincides with the normal to the quadric. Six of these points are the vertices of the quadric.*

It may also be verified that the above equation in λ cannot be an identity for a real quadric. In other words, *the line of striction of a real central quadric can in no case be a geodesic*—a fact which could be deduced from other considerations also.

From the above, it follows that the osculating plane to L^4 at a vertex of the quadric will be the plane containing the tangent at the vertex, and passing through the centre. It may now be verified that *the osculating planes of the curve at a pair of opposite vertices of the quadric form with the principal planes which pass through the axis joining the vertices a harmonic pencil of planes.*

Also, *the tangent to a line of striction at a vertex is parallel to the tangent to the other line of striction at the opposite vertex.*

4. The flex-points of either line of striction are given by the quartic in the syzygetic pencil apolar to the quartic determined by the principal plane, which in this case is the plane at infinity. The flex-points are therefore given by a quartic of the form $\lambda^4+k\lambda^2+1=0$ which is apolar to $\lambda^4+2A\lambda^2+1=0$. The flex-points of either curve are thus given by $\lambda^4-6\lambda^2/A+1=0$, and will be real* if $0<A<3$, i.e. if $A(A-3)$ is negative, i.e. if

$$(2b^{-2}+c^{-2}-a^{-2})\ (2a^{-2}+c^{-2}-b^{-2})>0.$$

It may, however, happen that the above quartic is a perfect square. $A\lambda^4-6\lambda^2+A$ will be a multiple of λ^2, $(\lambda^2-1)^2$, or $(\lambda^2+1)^2$ when $A=0$, 3, or -3. When this happens the flexes coincide in pairs with the extremities of an axis of the quadric, as

* They will be either all real or all imaginary, since the four points are the reflections of one another in the co-ordinate axes.

may also be seen from the symmetry of the flex-points about the axes. But it is well known in the geometry of R^4 that if two of the flex-points coincide at a point, this point is a point of inflexion. The necessary and sufficient condition that the line of striction should have two inflexions is therefore given by $A=0$, or $A=\pm 3$.* The inflexions are at the ends of the x-axis, the y-axis, or the z-axis according as $A=3$, $A=0$, or $A=-3$. Only the first two cases give real inflexions. Also when one line of striction has inflexions, the other line has inflexions at the same points.

This condition is also the condition that the tangents to the line of striction should belong to a linear complex, since an R^4 belonging to a linear complex has two points of inflexion, and conversely.

When the line of striction has two inflexions, the principal tetrahedron of the curve ceases to exist. There is now only one chord p_x^2 (see **1**). In the general case, in **1**, the chords p_x^2, q_x^2, r_x^2 are the axes of the quadric. The chord p_x^2 in the present case continues to be an axis.

The above condition admits of an interesting geometrical interpretation. We may without loss of generality take for the x-axis the *major axis* of the principal elliptic section. Hence $a>b$, and $A>0$. For real inflexions of the lines of striction, we must have therefore $A=3$. The real central circular sections of the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ are given by

$$\frac{y}{b}\sqrt{a^2-b^2}\pm\frac{z}{c}\sqrt{a^2+c^2}=0,$$

and are at right angles when $A=3$.

Hence, *the necessary and sufficient condition that either line of striction should have two real points of inflexion is that the real central circular sections of the quadric should be at right angles.*

Also, *if a line of striction has real inflexions, these will be the extremities of the major axis of the principal elliptic section.*

If the inflexions are imaginary, they lie on the z-axis, and the (imaginary) circular sections through the z-axis will be at right angles.

* Compare Sommerville, *loc. cit.* p. 305, Ex. 4.

The canonical equations of a quartic with two inflexions are

$$x:y:z:w=t^4:t^3:t:1.$$

The curve lies on a unique cubic scroll of the first species $xz^2=wy^2$ having the line joining the inflexions of the curve as a double line. The line $x=0$, $w=0$ is the simple directrix of the scroll, and is the polar line of $y=0$, $z=0$ with respect to the quadric through the curve. When the curve is a line of striction on the quadric, this line is at infinity. Also, it is well known that any generator of the scroll meets the curve in two points harmonically separating the points where the generator meets the two directrix lines. Hence again it follows that the simple directrix is at infinity, since from the symmetry of the curve about the axes, every bisecant of the curve meeting an axis is bisected by it.

SOME INFINITE INTEGRALS INVOLVING WEBER'S PARABOLIC CYLINDER FUNCTIONS

By R. S. VARMA, *Christ Church College, Cawnpore.*

[Received 20 December 1937]

1. The object of this paper is to evaluate some infinite integrals involving Weber's parabolic cylinder functions. Thus in **2**, an integral involving a Whittaker function $W_{k,m}(x)$ and a Kummer function ${}_1F_1$ is investigated. It will be seen that this general integral gives as a particular case Mayr's generalisation of the orthogonal property for Laguerre polynomials $L_n^m(x)$ and also gives a special case of Watson's integral, involving the product of two parabolic cylinder functions. In **3**, Operational Calculus is used to obtain from the integral of **2**, an infinite integral involving a Whittaker function $W_{k,m}(x)$ and a generalised hypergeometric function ${}_1F_2$. In **4**, it is shown that a particular case of the integral in **3** gives the following important theorem:

If
$$\phi(x) = x^{n+\frac{1}{2}} e^{-\frac{1}{4}x^2} D_{2n}(x)$$
and
$$\psi(x) = x^{n-\frac{1}{2}} e^{-\frac{1}{4}x^2} D_{2n+1}(x) \tag{1.1}$$
then $\phi(x)$ and $\psi(x)$ are J_n-transforms or skew J_n-transforms of each other, according as n is even or odd.* A similar theorem for the parabolic cylinder functions of negative orders has been recently established by me†. Incidentally we also discover in **5** two integral expressions for the Bateman function $k_{2n}(x)$. In **6**, an integral involving a parabolic cylinder function and a Bessel function $I_n(x)$ is evaluated. Finally in **7**, we deduce an integral involving the product of a parabolic cylinder function and two Bessel functions $J_n(x)$ and $I_n(x)$. It is interesting to note that this integral gives, as particular cases, integral expressions for the Humbert function‡ $J_{m,n}(x)$ defined by

$$J_{m,n}(x) = \frac{x^{m+n}}{3^{m+n}\Gamma(m+1)\,\Gamma(n+1)}\,{}_0F_2\left(m+1, n+1; -\frac{x^3}{27}\right)$$

* For the theory of these transforms see, E. C. Titchmarsh, *Introduction to the theory of Fourier integrals* (Oxford University Press, 1937).

† R. S. Varma, 'Extensions of some self-reciprocal functions', *Journal Indian Math. Soc.* (2), 2 (1937), 269-275.

‡ P. Humbert, 'Les fonctions de Bessel du troisieme ordre', *Atti-Pont. Accad. delle Scienze*, anno LXXXIII. Sess. 111 del 16 Febbraio 1930, 128-146.

2. Goldstein* has shown that

$$\int_0^\infty x^{l-1}e^{-(\alpha^2+\frac{1}{2})x}W_{k,m}(x)\,dx$$
$$=\frac{\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}\,{}_2F_1(l+m+\tfrac{1}{2},\,l-m+\tfrac{1}{2};\,l-k+1;\,-\alpha^2),$$

where $l\pm m+\frac{1}{2}>0$ and $I(\alpha)<1$. For $\alpha=0$, this gives

$$\int_0^\infty x^{l-1}e^{-\frac{1}{2}x}W_{k,m}(x)\,dx=\frac{\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}. \quad (2.1)$$

Now expanding ${}_1F_1$, we have

$$I = \int_0^\infty x^{l-1}e^{-\frac{1}{2}x}{}_1F_1(\alpha;\,\beta;\,-xt)W_{k,m}(x)\,dx$$
$$=\int_0^\infty \sum_{r=0}^{\infty}\frac{(\alpha,r)}{(\beta,r)r!}(-t)^r x^{l+r-1}e^{-\frac{1}{2}x}W_{k,m}(x)\,dx$$
$$=\sum_{r=0}^{\infty}\frac{(\alpha,r)(-t)^r}{(\beta,r)r!}\int_0^\infty x^{l+r-1}e^{-\frac{1}{2}x}W_{k,m}(x)\,dx$$

a process obviously justifiable,

$$=\sum_{r=0}^{\infty}\frac{(\alpha,r)(-t)^r\Gamma(l+r+m+\frac{1}{2})\,\Gamma(l+r-m+\frac{1}{2})}{(\beta,r)r!\,\Gamma(l+r-k+1)},$$

where we have used (2.1).

It follows therefore that

$$I=\frac{\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}\,{}_3F_2\left[\begin{matrix}\alpha,\,l+m+\frac{1}{2},\,l-m+\frac{1}{2}\\ \beta,\,l-k+1\end{matrix};-t\right] \quad (2.2)$$

$|t|<1$ and $l\pm m+\frac{1}{2}>0$.

We shall now give some particular cases of this general integral. Since

$${}_1F_1(-n;\,p+1;\,x)=\frac{n!\,\Gamma(p+1)}{\Gamma(p+n+1)}L_n^p(x)$$

we get from (2.2) true when $l\pm m+\frac{1}{2}>0$, $t=-1$, α a negative integer,

$$\int_0^\infty x^{l-1}e^{-\frac{1}{2}x}L_n^p(x)W_{k,m}(x)\,dx$$
$$=\frac{\Gamma(p+n+1)\,\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{n!\,\Gamma(p+1)\,\Gamma(l-k+1)}$$
$$\times{}_3F_2\left[\begin{matrix}-n,\,l+m+\frac{1}{2},\,l-m+\frac{1}{2}\\ p+1,\,l-k+1\end{matrix};1\right]$$

true when $l\pm m+\frac{1}{2}>0$ and n is a positive integer.

* S. Goldstein, 'Operational representation of Whittaker's confluent hypergeometric function and Weber's parabolic cylinder function', *Proc. Lond. Math. Soc.* (2) 34 (1932), 103-125.

This reduces further, by virtue of the relation,

$$L_m^q(x) = \frac{(-)^m}{m!}(x)^{-\frac{1}{2}(q+1)}e^{\frac{1}{2}x}W_{m+\frac{1}{2}q+\frac{1}{2},\frac{1}{2}q}(x)$$

to the following integral, due to K. Mayr,*

$$\int_0^\infty x^{l+\frac{1}{2}q-\frac{1}{2}}e^{-x}L_n^p(x)L_m^q(x)dx$$

$$= \frac{(-)^m\Gamma(p+n+1)\,\Gamma(l+\frac{1}{2}q+\frac{1}{2})\,\Gamma(l-\frac{1}{2}q+\frac{1}{2})}{n!\,m!\,\Gamma(p+1)\,\Gamma(l-m-\frac{1}{2}q+\frac{1}{2})}$$

$$\times {}_3F_2\left[\begin{matrix}-n,\ l+\frac{1}{2}q+\frac{1}{2},\ l-\frac{1}{2}q+\frac{1}{2}\\ p+1,\ l-m-\frac{1}{2}q+\frac{1}{2}\end{matrix};1\right] \qquad (2.3)$$

$$= \frac{(-)^m\Gamma(p+n+1)\,\Gamma(l+\frac{1}{2}q+\frac{1}{2})\,\Gamma(l-\frac{1}{2}q+\frac{1}{2})}{n!\,m!\,\Gamma(p+1)\,\Gamma(l+n-m-\frac{1}{2}q+\frac{1}{2})}$$

$$\times \frac{\Gamma(p+n-\frac{1}{2}q-m-l+\frac{1}{2})}{\Gamma(p-m-l-\frac{1}{2}q+\frac{1}{2})}\,{}_3F_2\left[\begin{matrix}-n,\ p-l-\frac{1}{2}q+\frac{1}{2},\ p-l+\frac{1}{2}q+\frac{1}{2}\\ p+1,\ p-m-l-\frac{1}{2}q+\frac{1}{2}\end{matrix};1\right],$$

where we have used Hardy's result,† viz.

$${}_3F_2\left[\begin{matrix}\alpha_1,\ \alpha_2,\ \alpha_3\\ \beta_1,\ \beta_2\end{matrix};1\right] = \frac{\Gamma(\beta_2)\,\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\beta_2-\alpha_3)\,\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2)}$$

$$\times {}_3F_2\left[\begin{matrix}\beta_1-\alpha_1,\ \beta_1-\alpha_2,\ \alpha_3\\ \beta_1,\ \beta_1+\beta_2-\alpha_1-\alpha_2\end{matrix};1\right].$$

The above integral can be considered to be the generalisation of the following property of orthogonality for Laguerre polynomials:

$$\int_0^\infty e^{-t}t^\alpha L_m^\alpha(t)L_n^\alpha(t)dt = \begin{cases} 0 & m\neq n \\ \dfrac{\Gamma(\alpha+n+1)}{n!} & m=n. \end{cases}$$

Since Laguerre polynomials reduce to parabolic cylinder functions of positive integral orders, the result (2.3) furnishes also the values of the integrals of the type

$$\int_0^\infty z^p D_m(z)D_n(z)dz, \qquad (2.4)$$

where m and n are integers.

* K. Mayr, 'Integraleigenschaften der Hermiteschen und Laguerreschen polynome', *Math. Zeit.* 39 (1935), pp. 597-604. A generalisation of Mayr's integral has recently been given by A. Erdelyi, 'Über einige bestimmte integrale, in denen die Whittakerschen $M_{k,m}$-Funktionen auftreten', *Math. Zeit.* 40 (1936), 693-702.

† G. H. Hardy, 'A chapter from Ramanujan's note book', *Proc. Camb. Phil. Soc.* 21 (1923), 492-503.

It may be remarked that a more general integral of the type (2.4) is the following due to Watson.*

$$\int_{\infty}^{(+0)} z^p D_m(z) D_n(z)\, dz = \pi^{\frac{3}{2}} (\sqrt{2})^{n-m-p-1} e^{\pi i(p-\frac{1}{2})}$$

$$\times\left[\frac{\Gamma(-\frac{1}{2}m)}{\Gamma(-m)\,\Gamma(-p)\,\Gamma(\frac{1}{2}p-\frac{1}{2}n+1)}\,{}_3F_2\left(\begin{matrix}-\frac{1}{2}m,\ \frac{1}{2}p+\frac{1}{2},\ \frac{1}{2}p+1\\ \frac{1}{2},\ 1+\frac{1}{2}p-\frac{1}{2}n\end{matrix};1\right)\right.$$

$$\left.+\frac{\Gamma(\frac{1}{2}-\frac{1}{2}m)}{\Gamma(-m)\Gamma(-p-1)\Gamma(\frac{1}{2}p-\frac{1}{2}n+\frac{3}{2})}\,{}_3F_2\left(\begin{matrix}\frac{1}{2}-\frac{1}{2}m,\ \frac{1}{2}p+1,\ \frac{1}{2}p+\frac{3}{2}\\ \frac{3}{2},\ \frac{3}{2}+\frac{1}{2}p-\frac{1}{2}n\end{matrix};1\right)\right].$$

3. In (2.2), put $t=1/p$ where

$$\frac{1}{p^{\alpha}} \doteqdot \frac{u^{\alpha}}{\Gamma(\alpha+1)} \qquad R(\alpha+1)>0. \tag{3.1}$$

We then have

$$\int_0^{\infty} x^{l-1} e^{-\frac{1}{2}x}\,{}_1F_1\left(\alpha;\beta;-\frac{x}{p}\right) W_{k,m}(x)\,dx$$

$$=\frac{\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}\,{}_3F_2\left[\begin{matrix}\alpha,\ l+m+\frac{1}{2},\ l-m+\frac{1}{2}\\ \beta,\ l-k+1\end{matrix};-\frac{1}{p}\right]. \tag{3.2}$$

Now

$${}_1F_1\left(\alpha;\beta;-\frac{x}{p}\right)=\sum_{r=0}^{\infty}\frac{(\alpha,r)}{r!\,(\beta,r)}\left(-\frac{x}{p}\right)^r$$

$$\doteqdot \sum_{r=0}^{\infty}\frac{(\alpha,r)}{r!\,(\beta,r)}\,\frac{(-xu)^r}{\Gamma(r+1)}$$

$$= {}_1F_2(\alpha;\beta,1;-xu), \tag{3.3}$$

and similarly

$${}_3F_2\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{matrix};-\frac{1}{p}\right)\doteqdot {}_3F_3\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\ 1,\beta_1,\beta_2\end{matrix};-u\right). \tag{3.4}$$

Hence finding the original of both the sides of (3.2) by the help of (3.3) and (3.4), we obtain that

$$\int_0^{\infty} x^{l-1} e^{-\frac{1}{2}x}\,{}_1F_2(\alpha;\beta,1;-xu)\,W_{k,m}(x)\,dx$$

$$=\frac{\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}\,{}_3F_3\left[\begin{matrix}\alpha,\ l+m+\frac{1}{2},\ l-m+\frac{1}{2}\\ 1,\ \beta,\ l-k+1\end{matrix};-u\right]. \tag{3.5}$$

* G. N. Watson, 'The harmonic functions associated with the parabolic cylinder', *Proc. Lond. Math. Soc.* (2), 8 (1910), 393-421.

Since

$$J_n(x)=\frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n {}_0F_1(1+n;-\tfrac{1}{4}x^2)$$

the above integral reduces for $\alpha=1$ to

$$\int_0^\infty s^{2l-n-1}e^{-\frac{1}{4}s^2}J_n(sa)W_{k,m}(\tfrac{1}{2}s^2)ds$$

$$=\frac{2^{l-n-1}a^n\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(l-k+1)}\,{}_2F_2\left[\begin{matrix}l+m+\frac{1}{2},\,l-m+\frac{1}{2}\\ n+1,\,l-k+1\end{matrix};-\tfrac{1}{2}a^2\right]. \tag{3.6}$$

In (3.6) if we put $m=\frac{1}{4}$, we obtain the integral

$$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2}D_\mu(x)J_n(ax)dx$$

$$=\frac{\sqrt{\pi}\,a^n\Gamma(\nu+n+1)}{2^{\frac{3}{2}n+\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{2}}\Gamma(n+1)\,\Gamma(\frac{1}{2}\nu+\frac{1}{2}n-\frac{1}{2}\mu+1)}$$

$$\times{}_2F_2\left[\begin{matrix}\frac{1}{2}\nu+\frac{1}{2}n+\frac{1}{2},\,\frac{1}{2}\nu+\frac{1}{2}n+1\\ n+1,\,\frac{1}{2}\nu+\frac{1}{2}n-\frac{1}{2}\mu+1\end{matrix};-\tfrac{1}{2}a^2\right],\ \nu+n+1>0 \tag{3.7}$$

obtained by me elsewhere*.

4. The generalised hypergeometric function occurring on the right side of (3.7) reduces to Kummer's function in four cases.

CASE 1. When $\nu=n+1$.

We have

$$\int_0^\infty x^{n+1}e^{-\frac{1}{4}x^2}D_\mu(x)J_n(ax)dx$$

$$=\frac{2^{\frac{1}{2}\mu}a^n\Gamma(n+\frac{3}{2})}{\Gamma(n-\frac{1}{2}\mu+\frac{3}{2})}\,{}_1F_1(n+\tfrac{3}{2};\,n-\tfrac{1}{2}\mu+\tfrac{3}{2};-\tfrac{1}{2}a^2)$$

$$=\frac{2^{\frac{1}{2}\mu}a^n\,\Gamma(n+\frac{3}{2})}{\Gamma(n-\frac{1}{2}\mu+\frac{3}{2})}\,e^{-\frac{1}{2}a^2}{}_1F_1(-\tfrac{1}{2}\mu:\,n-\tfrac{1}{2}\mu+\tfrac{3}{2};\,\tfrac{1}{2}a^2), \tag{4.1}$$

where we have used Kummer's first transformation formula, viz.

$${}_1F_1(\alpha;\rho;z)=e^z{}_1F_1(\rho-\alpha;\rho;-z).$$

* R. S. Varma, 'Some infinite integrals involving parabolic cylinder Functions'. This paper has been accepted for publication in the part of *Journal de Mathematique* dedicated to Prof. J. Hadamard.

Now we know that, for all values of n,

$$D_n(x)=\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}n)}2^{\frac{1}{2}n}e^{-\frac{1}{4}x^2}{}_1F_1(-\tfrac{1}{2}n:\tfrac{1}{2};\tfrac{1}{2}x^2)$$
$$+\frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{1}{2}n)}2^{\frac{1}{2}n-\frac{1}{2}}xe^{-\frac{1}{4}x^2}{}_1F_1(\tfrac{1}{2}-\tfrac{1}{2}n;\tfrac{3}{2};\tfrac{1}{2}x^2)$$

which reduces, when n is an integer, to

$$D_{2n}(x)=\frac{\Gamma(\frac{1}{2})2^n e^{-\frac{1}{4}x^2}}{\Gamma(-n+\frac{1}{2})}{}_1F_1(-n;\tfrac{1}{2};\tfrac{1}{2}x^2)$$

and

$$D_{2n+1}(x)=\frac{\Gamma(-\frac{1}{2})2^n e^{-\frac{1}{4}x^2}}{\Gamma(-n-\frac{1}{2})}{}_1F_1(-n;\tfrac{3}{2};\tfrac{1}{2}x^2).$$

For $\mu=2n$, (4.1) therefore gives

$$\int_0^\infty x^{n+\frac{1}{2}}e^{-\frac{1}{4}x^2}D_{2n}(x)\sqrt{ax}\,J_n(ax)\,dx=(-)^n a^{n-\frac{1}{2}}e^{-\frac{1}{4}a^2}D_{2n+1}(a) \tag{4.2}$$

and for $\mu=2n+2$ it reduces to

$$\int_0^\infty x^{n+\frac{1}{2}}e^{-\frac{1}{4}x^2}D_{2n+2}(x)\sqrt{ax}\,J_n(ax)\,dx$$
$$=(-)^{n+1}a^{n+\frac{1}{2}}e^{-\frac{1}{4}a^2}D_{2n+2}(a). \tag{4.3}$$

CASE 2. When $\mu=1$.

Since

$$D_1(x)=xe^{-\frac{1}{4}x^2}$$

we get a known integral in the theory of Bessel functions.

CASE 3. When $\nu=n$.

We have

$$\int_0^\infty x^n e^{-\frac{1}{4}x^2}D_\mu(x)J_n(ax)\,dx$$
$$=\frac{\sqrt{\pi}\,a^n\Gamma(2n+1)}{2^{2n-\frac{1}{2}\mu+\frac{1}{2}}\Gamma(n+1)\,\Gamma(n-\frac{1}{2}\mu+1)}{}_1F_1(n+\tfrac{1}{2};n-\tfrac{1}{2}\mu+1;-\tfrac{1}{2}a^2).$$

Here as in case 1, we obtain for $\mu=2n-1$,

$$\int_0^\infty x^{n-\frac{1}{2}}e^{-\frac{1}{4}x^2}D_{2n-1}(x)\sqrt{ax}\,J_n(ax)\,dx$$
$$=(-)^{n-1}a^{n-\frac{1}{2}}e^{-\frac{1}{4}a^2}D_{2n-1}(a) \tag{4.4}$$

and for $\mu = 2n+1$,

$$\int_0^\infty x^{n-\frac{1}{2}} e^{-\frac{1}{4}x^2} D_{2n+1}(x) \sqrt{ax}\, J_n(ax)\, dx = (-)^n a^{n+\frac{1}{2}} e^{-\frac{1}{4}a^2} D_{2n}(a). \tag{4.5}$$

CASE 4. When $\mu = 0$.

Here also we get a known integral involving Bessel functions.

Now the integrals (4.2) and (4.5) at once lead us to the important theorem enunciated in **1.** The integrals (4.3) and (4.4) have recently been deduced by Dr. S. C. Mitra* and in the language of transforms can be read as follows:

The functions

$$f_1(x) = x^{n+\frac{1}{2}} e^{-\frac{1}{4}x^2} D_{2n+2}(x)$$

and

$$f_2(x) = x^{n-\frac{1}{2}} e^{-\frac{1}{4}x^2} D_{2n-1}(x)$$

are self-reciprocal or skew-reciprocal in the Hankel transform according as n is an integer, odd or even.

5. Since Bateman's function $k_{2n}(x)$ is given by

$$k_{2n}(x) = (-)^{n-1} 2xe^{-x} {}_1F_1(1-n; 2; 2x)$$

(3.7) gives, as particular cases, the following integral expressions for the Bateman function:

$$\int_0^\infty x^{n+\frac{1}{2}} e^{-\frac{1}{4}x^2} D_{2n-1}(x) J_{n+\frac{1}{2}}(ax)\, dx$$

$$= (-)^{n-1} \Gamma(n+1) 2^n a^{n-\frac{3}{2}} e^{-\frac{1}{4}a^2} k_{2n}(\tfrac{1}{4}a^2).$$

6. Now multiply both sides of (3.7) by $e^{-a^2y^2}$ and integrate with respect to a between the limits 0 and ∞. The left side of (3.7) then becomes

$$\int_0^\infty e^{-y^2a^2} da \int_0^\infty x^\nu e^{-\frac{1}{4}x^2} D_\mu(x) J_n(ax)\, dx$$

which, on changing the order of integration,—a process obviously permissible—and then using the known integral†

$$\int_0^\infty \exp(-y^2t^2) J_{2\nu}(at)\, dt = \frac{\sqrt{\pi}}{2y} \exp\left(-\frac{a^2}{8y^2}\right) I_\nu\left(\frac{a^2}{8y^2}\right)$$

* S. C. Mitra, 'On parabolic cylinder functions which are self-reciprocal in the Hankel transform', *Math, Zeit.* 43 (1937), 205-211.

† Watson, *Bessel Functions*, p. 394.

reduces to

$$\frac{\sqrt{\pi}}{2y}\int_0^\infty x^\nu e^{-\frac{1}{4}x^2-\frac{x^2}{8y^2}} D_\mu(x) I_{\frac{1}{2}n}\left(\frac{x^2}{8y^2}\right)dx.$$

Now it is easy to prove that

$$\int_0^\infty a^l e^{-y^2a^2} {}_2F_2\left(\begin{matrix}\alpha_1, \alpha_2\\ \beta_1, \beta_2\end{matrix}; -\tfrac{1}{2}a^2\right)da$$
$$= \frac{\Gamma(\frac{1}{2}l+\frac{1}{2})}{2y^{l+1}}\, {}_3F_2\left[\begin{matrix}\alpha_1, \alpha_2, \frac{1}{2}l+\frac{1}{2}\\ \beta_1, \beta_2\end{matrix}; -\frac{1}{2y^2}\right].$$

By the help of this, the right side of (3.7) yields

$$\frac{\sqrt{\pi}\,\Gamma(\nu+n+1)\,\Gamma(\frac{1}{2}n+\frac{1}{2})}{y^{n+1}2^{\frac{3}{2}n+\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{3}{2}}\Gamma(n+1)\,\Gamma(\frac{1}{2}\nu+\frac{1}{2}n-\frac{1}{2}\mu+1)}$$
$$\times {}_3F_2\left(\begin{matrix}\frac{1}{2}\nu+\frac{1}{2}n+\frac{1}{2}, \frac{1}{2}\nu+\frac{1}{2}n+1, \frac{1}{2}n+\frac{1}{2}\\ n+1, \frac{1}{2}\nu+\frac{1}{2}n-\frac{1}{2}\mu+1\end{matrix}; -\frac{1}{2y^2}\right).$$

It follows, therefore, that

$$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2\left(1+\frac{1}{2y^2}\right)} D_\mu(x) I_{\frac{1}{2}n}\left(\frac{x^2}{8y^2}\right)dx$$
$$= \frac{\Gamma(\nu+n+1)\,\Gamma(\frac{1}{2}n+\frac{1}{2})}{y^n 2^{\frac{3}{2}n+\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{2}}\Gamma(n+1)\,\Gamma(\frac{1}{2}\nu+\frac{1}{2}n-\frac{1}{2}\mu+1)}$$
$$\times {}_3F_2\left[\begin{matrix}\frac{1}{2}\nu+\frac{1}{2}n+\frac{1}{2}, \frac{1}{2}\nu+\frac{1}{2}n+1; \frac{1}{2}n+\frac{1}{2}\\ n+1, \frac{1}{2}\nu+\frac{1}{2}n-\frac{1}{2}\mu+1\end{matrix}; -\frac{1}{2y^2}\right], \nu+n+1>0.$$

7. Finally in (3.7) let us put $a=1/p$ where, as in **3**,

$$\frac{1}{p^\alpha} \doteqdot \frac{t^\alpha}{\Gamma(\alpha+1)} \quad R(\alpha)>-1.$$

Then

$$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2} D_\mu(x) J_n\left(\frac{x}{p}\right)dx$$
$$= \frac{\sqrt{\pi}\,\Gamma(\nu+n+1)}{2^{\frac{3}{2}n-\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}}\Gamma(n+1)\,\Gamma(\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{2}n+1)}$$
$$\times \frac{1}{p^n}\, {}_2F_2\left[\begin{matrix}\frac{1}{2}\nu+\frac{1}{2}n+\frac{1}{2}, \frac{1}{2}\nu+\frac{1}{2}n+1\\ n+1, \frac{1}{2}\nu+\frac{1}{2}n-\frac{1}{2}\mu+1\end{matrix}; -\frac{1}{2p^2}\right]. \tag{7.1}$$

Now we shall find the original of both the sides of (7.1). Since*

* S. C. Mitra, 'On certain integrals and expansions involving Bessel functions', *Bulletin Calcutta Math. Soc.* 25 (1933), 81-98.

$$J_n\left(\frac{x}{p}\right) \doteqdot J_n(\sqrt{2xt})I_n(\sqrt{2xt}),$$

the original of the left hand side of (7.1) is

$$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2} D_\mu(x) J_n(\sqrt{2xt}) I_n(\sqrt{2xt})\, dx.$$

Following the method of **3**, it is easy to show that

$$\frac{1}{p^n}\, {}_2F_2\left[\begin{matrix} m_1, m_2 \\ n_1, n_2 \end{matrix}; -\frac{1}{2p^2}\right] \doteqdot \frac{t^n}{\Gamma(n+1)}\, {}_2F_4\left[\begin{matrix} m_1, m_2 \\ n_1, n_2, \frac{1}{2}n+\frac{1}{2}, \frac{1}{2}n+1 \end{matrix}; -\frac{t^2}{8}\right].$$

It follows therefore that the original of the right hand side of (7.1) is

$$\frac{\sqrt{\pi}\,\Gamma(\nu+n+1)t^n}{2^{\frac{3}{2}n-\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}}\left\{\Gamma(n+1)\right\}^2 \Gamma(\frac{1}{2}\nu+\frac{1}{2}n-\frac{1}{2}\mu+1)}$$

$$\times\, {}_2F_4\left[\begin{matrix} \frac{1}{2}\nu+\frac{1}{2}n+\frac{1}{2}, \frac{1}{2}\nu+\frac{1}{2}n+1 \\ n+1, \frac{1}{2}\nu+\frac{1}{2}n-\frac{1}{2}\mu+1, \frac{1}{2}n+\frac{1}{2}, \frac{1}{2}n+1 \end{matrix}; -\frac{t^2}{8}\right].$$

Hence we deduce from (7.1) the result

$$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2} D_\mu(x) J_n(\sqrt{2xt}) I_n(\sqrt{2xt})\, dx$$

$$= \frac{\sqrt{\pi}\,\Gamma(\nu+n+1)t^n}{2^{\frac{3}{2}n-\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}}\left\{\Gamma(n+1)\right\}^2 \Gamma(\frac{1}{2}\nu+\frac{1}{2}n-\frac{1}{2}\mu+1)}$$

$$\times\, {}_2F_4\left[\begin{matrix} \frac{1}{2}\nu+\frac{1}{2}n+\frac{1}{2}, \frac{1}{2}\nu+\frac{1}{2}n+1 \\ n+1, \frac{1}{2}\nu+\frac{1}{2}n-\frac{1}{2}\mu+1, \frac{1}{2}n+\frac{1}{2}, \frac{1}{2}n+1 \end{matrix}; -\frac{t^2}{8}\right]$$

$$\nu+n+1>0. \qquad (7.2)$$

Since the generalised hypergeometric function on the right of (7.2) reduces to ${}_0F_2$ under special cases, the result (7.2) gives integral expressions for the Humbert function $J_{m,n}(x)$.

ON LINEAR COMPLEXES OF MINIMUM RANK CONTAINING ALL THE TANGENTS OF A RATIONAL NORM CURVE*

By B. RAMAMURTI, *Annamalai University.*

[Received 23 November 1937]

1. The rational curve $R^n(3)$ of order n in [3], whose tangents belong to a linear complex, has been studied in detail by several writers. Picard† showed that such a curve possesses in general $2n-6$ points of inflexion. G. H. Grace‡ attempted to prove the converse, that if an $R^n(3)$ has $2n-6$ inflexions its tangents belong to a linear complex. The proof was however faulty, and M. F. Egan§ gave an example of a sextic curve for which the converse theorem is not true.

These problems can also be interpreted and approached from another point of view which affords greater insight into them. Any complex curve‖ $R^n(3)$ is the projection of a rational norm curve R^n in $[n]$ from the singular $[n-4]$ of a linear complex L of rank 4, which contains all the tangents of R^n. The geometry of $R^n(3)$ is identical with the geometry of $R^n(n)$ in relation to the vertex space $[n-4]$, which in turn is identical with the invariant theory of the linear ∞^3 system of binary n-ics I_n^3 determined on R_n by the primes through the $[n-4]$. The object of this paper is to correlate the results from these points of view. We first study linear complexes of the lowest rank 4, which contain the tangents of R^n. Theorems 3 and 4 are equivalent to Picard's theorem while Theorems 6 and 7 correspond to the conclusions that may be drawn as regards the converse of Picard's theorem. It will be interesting if Theorems 6 and 7 could be proved directly without the aid of the results in [3].

* I take this opportunity to thank Dr. C. N. Srinivasiengar for drawing my attention to the references cited below, which led me to the present investigation, and for his kindness in lending me a reprint of Egan's paper.

† *Encycl. der. Math. Wiss.* III$_2$, Heft 8, 1024.

‡ *Proc. Camb. Phil. Soc.* 11 (1902), 132.

§ *Proc. Roy. Irish Academy*, 29 (1911), 62.

‖ A curve whose tangents belong to a linear complex will be called a complex curve.

2. The rank of a linear complex

$$\Sigma a_{rs}(x_r y_s - x_s y_r) = 0$$

is always even, since the matrix $||a_{rs}||$ is skew-symmetric. Let us take a linear complex L of rank 2 in $[n]$. It consists of all lines intersecting a $[n-2]$. If $u_{1x}=0$, $u_{2x}=0$ be two primes through the $[n-2]$, the equation of L is

$$u_{1x}u_{2y} - u_{1y}u_{2x} = 0. \qquad (1)$$

Let the primes cut out on R^n point-sets given parametrically by the binary forms a_{1x}^n and a_{2x}^n. The points x and y on R^n are conjugate for L, i.e. for the null-system defined by L, if

$$A_{12}(x, y) \equiv a_{1x}^n a_{2y}^n - a_{1y}^n a_{2x}^n = 0. \qquad (2)$$

Expanding in Gordan's series in powers of (xy), the leading term is

$$(xy)\left\{ (a_1 a_2) a_{1x}^{n-1} a_{2x}^{n-1} \right\}_{y^{n-1}}, \qquad (3)$$

the subscript denoting the $(n-1)$th polar with respect to y.

Therefore L contains the tangents of R^n at the points given by the Jacobian of a pencil of n-ics. L will contain all the tangents of R^n only if the Jacobian vanishes identically. This is impossible as the forms a_{1x}^n, a_{2x}^n are distinct. Therefore,

there is no linear complex of rank 2 containing all the tangents of R^n.

Let us now consider a linear complex of rank 4 so that the singular region is a $[n-4]$. It can be expressed in the form

$$(u_{1x}u_{2y} - u_{1y}u_{2x}) - (u_{3x}u_{4y} - u_{3y}u_{4x}) = 0, \qquad (4)$$

where the $[n-2]$'s corresponding to the pairs of primes u_1u_2, u_3u_4 both contain the $[n-4]$ and are conjugate to L. The correspondence which the complex determines on R^n is

$$A_{12}(xy) - A_{34}(xy) = 0, \qquad (5)$$

and hence the complex contains the tangents of R^n at the points

$$(a_1a_2)a_{1x}^{n-1}a_{2x}^{n-1} - (a_3a_4)a_{3x}^{n-1}a_{4x}^{n-1} = 0. \qquad (6)$$

If L contains all the tangents of R^n, (6) must vanish identically. The two pencils must have the same Jacobian, or geometrically the two $[n-2]$'s must intersect the same $2n-2$ tangents of R^n. Let us call such $[n-2]$'s *associated* $[n-2]$'s. We have

THEOREM 1. *The minimum rank of a linear complex containing all the tangents of R^n is 4. The singular $[n-4]$ of such a complex must be the intersection of two associated $[n-2]$'s.*

It is well known that L can be thrown into the form (4) in ∞^4 ways. We have only to take any $[n-2]$ through the $[n-4]$, and this fixes uniquely the conjugate $[n-2]$ which also contains the $[n-4]$. More generally, let u_1, u_2, u_3, u_4 be any four primes through the $[n-4]$; the equation of any linear complex having the $[n-4]$ as its singular region is

$$\Sigma a_{rs}(u_{rx}u_{sy}-u_{sx}u_{ry}) = 0, \; r, s = 1, 2, 3, 4, r \neq s. \tag{7}$$

This contains the tangents of R^n at the points on R^n given by

$$\Sigma a_{rs}(a_r a_s) a_{rx}^{n-1} a_{sx}^{n-1} = 0. \tag{8}$$

If L contains all the tangents of R^n, (8) must vanish identically. There must therefore be a linear relation between the six Jacobians.

THEOREM 2. *With reference to a R^n in $[n]$, any $[n-4]$ represents a I_n^3 of binary n-ics. The $[n-4]$ is the singular region of a linear complex containing all the tangents of R^n if and only if the Jacobians of the pencils in I_n^3 are linearly dependent.*

3. We shall next prove

THEOREM 3. *The singular $[n-4]$ of a linear complex of rank 4 containing all the tangents of R^n intersects $2n-6$ osculating planes of R^n.*

Taking the correspondence (5) determined by such a complex, and expanding in Gordan's series, the leading term becomes in virtue of the vanishing of (6)

$$(xy)^3[(a_1a_2)^3 a_{1x}^{n-3} a_{2x}^{n-3} - (a_3a_4)^3 a_{3x}^{n-3} a_{4x}^{n-3}]_{y^{n-3}}, \tag{8}$$

the subscript denoting the $(n-3)$th polar with respect to y. If A_x^{2n-6} denote the form within the brackets, it is easily seen that L contains all the lines in the osculating planes at the points on R^n given by $A_x^{2n-6}=0$*. These osculating planes must intersect the singular $[n-4]$ of L. If not, let us consider one of the osculating planes. Since this has no point in common with the $[n-4]$, the two determine a $[n-1]$. Let P be a point not lying on the $[n-1]$ and Q a point in the osculating plane conjugate to P for L. Then Q is conjugate to every point in the osculating plane, to every point in the $[n-4]$, and to P, and hence to every point in $[n]$. Thus Q is a singular point of L which is contrary to the hypothesis. Hence the theorem follows.

*If the correspondence determined by L has the leading term $(xy)^{2\lambda+1}A_x^{n-2\lambda+1}A_y^{n-2\lambda+1}$, it is known that L contains all the osculating $[\lambda]$'s of R^n and the osculating $[\lambda+1]$'s at the points $A_x^{2n-4\lambda+2}$.

In general, any $[n-4]$ will intersect $4(n-3)$ osculating $[3]$'s, the points of contact corresponding to the Jacobian of the I_n^3 of binary forms† determined on R^n by the primes through the $[n-4]$. The intersection of an osculating $[2]$ with the $[n-4]$ is equivalent to the intersection of two consecutive osculating $[3]$'s, so that the point of contact corresponds to a double root of the Jacobian. Hence, if the $[n-4]$ intersects $2n-6$ osculating planes, the Jacobian must be a perfect square. Thus we have corresponding to Theorem 3, the following theorem in binary forms.

THEOREM 4. *If an I_n^3 contains two co-Jacobian pencils, in other words, if the Jacobians of the several pencils in the I_n^3 are linearly related, the Jacobian of I_n^3 is a perfect square.**

4. Given $2n-6$ osculating planes of R^n, it is easily seen from dimensional considerations that there are a finite number of $[n-4]$'s intersecting them. *Is every one of them the singular $[n-4]$ of a linear complex of rank 4 containing all the tangents of R^n?* The equivalent question in binary forms is the following: *If the Jacobian of a I_n^3 is a perfect square, are the Jacobians of the pencils in it linearly related, in other words, does it contain two co-Jacobian pencils?*

A direct answer seems to be difficult. We shall now correlate the above with known results about R^n in $[3]$ whose tangents belong to a linear complex, and show that the above is not true when $n>5$.

An R^n in $[n]$ projects from a $[n-4]$ not intersecting it, on a $[3]$ which does not intersect the $[n-4]$, into a rational curve of the same order. If the primes through the $[n-4]$ cut out on R^n point-sets given by a_{rx}^n, $r=1, 2, 3, 4$, the equations of $R^n(3)$ may be taken as

$$X_i=a_{ix}^n, \quad r=1, 2, 3, 4. \tag{9}$$

In general, any tangent and any osculating plane of $R^n(n)$ will project into a tangent and an osculating plane of $R^n(3)$. The $4(2n-3)$ osculating $[3]$'s which intersect the $[n-4]$ will project into $4(2n-3)$ osculating planes of $R^n(3)$ having contact of the third order with the curve.

If the $[n-4]$ is the singular region of a linear complex containing all the tangents of R^n, the tangents of $R^n(3)$ will also

† The Jacobian of a linear ∞^k system of n-ics is the form of order $(k+1)(n-k)$ whose roots are the roots of multiplicity $k+1$ of members of the system.

* This theorem has been proved by a different method by D. Hilbert, *Math. Annalen*, 33, 229.

belong to it. For, taking any tangent of R^n, it will determine with the $[n-4]$ a $[n-2]$ whose lines all belong to L, and the corresponding tangent of $R^n(3)$ is the intersection of this $[n-2]$ with $[3]$. Conversely, if the tangents of $R^n(3)$ belong to a linear complex L' in $[3]$, we can build a linear complex L in $[n]$ for which the $[n-4]$ is the singular region, and whose section by $[3]$ is L'; L will contain all the tangents of R^n.*

By Theorem 3, there are $2n-6$ osculating planes intersecting the $[n-4]$. Each of these will project into an inflexional tangent of $R^n(3)$. Hence we have Picard's theorem,

THEOREM 5. *A rational curve of order n in $[3]$ whose tangents belong to a linear complex has, in general, $2n-6$ points of inflexion.*

Grace attempted the converse problem by taking equations (9) of a $R^n(3)$ and considering the Jacobian of the ∞^5 systems determined by the Jacobians of the forms a_{1x}^n taken two by two. He argued that a point of inflexion is to be counted as a root of multiplicity 6 for the Jacobian, and hence the $2n-6$ points of inflexion give $6(2n-6)$ roots of J. But J is of order $6(2n-7)$. Hence $J\equiv 0$, which leads, as in Theorem 2, to show that the tangents belong to a linear complex.

The mistake in this argument was pointed out by M. F. Egan, who showed by an example that a point of inflexion is to be counted only five times amongst the roots of the Jacobian. Hence there are $(2n-12)$ roots more to be accounted for when $n \geqslant 6$. The proof therefore fails when $n \geqslant 6$. In fact, Egan has given an example of a rational sextic with six inflexions whose tangents do not belong to a linear complex†.

We have in $[n]$ and in binary forms the following

THEOREM 6. *In $[n]$ when $n>5$, every $[n-4]$ intersecting $2n-6$ osculating planes need not necessarily be the singular region of a linear complex of rank 4 containing all the tangents of R^n; in other words it need not be the intersection of two $[n-2]$'s intersecting the same $2n-2$ tangents of R^n.*

* From Theorem I it follows that the equation of a complex curve $R^n(3)$ can be expressed in the form $X_i=a_{ix}^n$, $i=1, 2, 3, 4$, where

$$\left(a_{1x}^n, a_{2x}^n\right)^1=\left(a_{3x}^n, a_{4x}^n\right)^1.$$

Compare Darboux, *Ann. Ecole. Normale*, (3) 7, 332; *Encycl. der. Math. Wiss.* III C_9, 1361.

† Egan, *loc. cit.* p. 51.

THEOREM 7. *If the Jacobian of a I_n^3 of binary forms is a perfect square, the Jacobians of the pencils in I_n^3 need not be linearly dependent, in other words I_n^3 need not contain two co-Jacobian pencils.*

It will be interesting if Theorem 6 could be established directly. In this connection, Gherardelli* has shown that there are two distinct types of planes in [6] intersecting six osculating planes, the planes of the first type lying on the equivalent quadric of R^6, and those of the other type not lying on it. From the first type of planes, R^6 is projected into a R^6 in [3] having six inflexions, but the tangents do not belong to a linear complex; but if it is projected from a plane of the second type, the tangents belong to a linear complex.

5. I had used the following argument to point out the mistake in Grace's paper, before my attention was drawn to the paper by Egan.

Let $P(\alpha)$ be the point of contact with R^n of the osculating plane intersecting the $[n-4]$ at Q, and let J be the Jacobian of the I_n^3 determined on R^n by the primes through the $[n-4]$. We have

LEMMA 1. *If the Jacobian of a pencil in I_n^3 contains α, it must contain α^2.*

A pencil in I_n^3 corresponds to primes through a $[n-2]$ containing $[n-4]$, and the Jacobian of the pencil gives the points on R^n, the tangents at which intersect the $[n-2]$. If the Jacobian of the pencil contains α, the $[n-2]$ must intersect the tangent at α, say at the point R. Then QR lies in the $[n-2]$ and also in the osculating plane at α. Hence the consecutive tangent also intersects the $[n-2]$ at α. Thus α is a double root of the Jacobian.

LEMMA 2. *If every member of the I_{2n-2}^5 determined by the Jacobians of the pencils in I_n^3 containing α has α as a double root, α is a root of multiplicity 5 of the Jacobian of I_{2n-2}^5.*

For there must be a I_{2n-2}^4 containing α, and hence α^2. We can therefore find a form $\alpha^{2n-3}\beta$ in the apolar system I_{2n-2}^{2n-8}, and the Jacobian of this apolar system will contain α as a root of multiplicity 5. But this Jacobian is also the Jacobian of the original system. Hence the result.

* *Rendiconti dell Accademia dei Lincei,* (6), 2 (1930), 173-79. Vide Wylie, *Amer. Journ. Math.* 57 (1935), 937-942. Gherardelli's paper is not accessible to me.

We have thus arrived at Egan's result that a point of inflexion counts five times in the Jacobian.

6. The following remarks on two statements which appear elsewhere are based on the ideas of this paper.

(*A*) "Both the singular spaces $P_{n-\lambda-1}$, $Q_{n-\lambda-1}$ of a correlation of *wertigkeit* λ and least rank $\lambda+1$ are secant spaces of the rational norm curve, each intersecting the curve in $n-\lambda$ points."*

With reference to a rational norm curve R^n in $[n]$, a correlation is said to be of *wertigkeit* λ, if out of the n points of R^n which correspond in the correlation to any point of R^n, λ coincide with that point. The null-system defined by a linear complex containing all the tangents of R^n is of wertigkeit 3. Hence, if it is of minimum rank 4, its singular $[n-4]$ must be a secant $[n-4]$ of R^n. But we may start with a $[n-2]$ which does not contain a secant $[n-4]$, and the resulting intersection of two *associated* $[n-2]$'s need not be a secant $[n-4]$. Thus, while every secant $[n-4]$ is a singular $[n-4]$ of a linear complex of rank 4 containing all the tangents, the converse is not true. It may however be remarked that this does not affect the other results in the paper cited.

(*B*) "Any P_{n-1} which intersects $\frac{n(n+1)}{2}$ osculating regions of $n-1$ dimensions of a norm curve N_{2n} is a generating region of the equivalent quadric"†.

When $n=3$, this becomes: Any plane which meets six osculating planes of a norm curve R^6 is a generating plane of the equivalent quadric.

As pointed out earlier, Gherardelli has proved that there are two families of planes intersecting six osculating planes of R^6, those of the first family not lying on the equivalent quadric, and those of the second lying on it. Hence the above theorem and Theorem (a') of the paper cited are not true, at any rate when $n=3$. The remaining theorems of the paper are not affected by this.

* R. Vaidyanathaswamy, 'Die Geometrie der rationalen Norm Curve', *Tohoku Math. Journ.* 32, 312-32, Theorem 6.

† R. Vaidyanathaswamy, 'Null pencil of Binary Quartics', *Proc. Lond. Math. Soc.* (2) 23, 317-325.

NOTES ON DENUMERANTS

By E. T. BELL, *California.*

[Received 27 February 1938]

1. *A Characteristic Equation.* If $n, a_1, \ldots, a_r$ are given integers, $n \geqslant 0$, $a_1 > 0, \ldots, a_r > 0$, the number of sets $(x_1, \ldots, x_r)$ of integers $x_1 \geqslant 0, \ldots, x_r \geqslant 0$ such that

$$a_1 x_1 + \ldots + a_r x_r = n$$

is (in Sylvester's terminology) the denumerant $A(n; a_1, \ldots, a_r)$, or simply $A(n)$. If x is not an integer $\geqslant 0$, $A(x)$ is defined to be zero. Let $e(0) = 1$, $e(x) = 0$, $x \neq 0$; and let $\sum_p$ refer to all selections of p distinct members of the set $a_1, \ldots, a_r$. Thus

$$\sum\nolimits_2 F(n - a_i - a_j)$$

refers to the $r(r-1)/2$ pairs $a_1, a_2; \ldots; a_1, a_r; \ldots; a_{r-1}, a_r$; etc.

Let $F(x) = 0$ if x is not an integer $\geqslant 0$, and let $F(x)$ take integer values > 0 for integer values $\geqslant 0$ of x. We shall show that $F(x) = A(x)$ is the unique solution of

$$F(x) - \sum\nolimits_1 F(x - a_i) + \sum\nolimits_2 F(x - a_i - a_j)$$
$$- \sum\nolimits_3 F(x - a_i - a_j - a_k) + \ldots + (-1)^r F(x - a_1 - \ldots - a_r) = e(x). \tag{1.1}$$

If x is not an integer $\geqslant 0$, (1.1) is identically satisfied, and $F(x) = A(x)$ is a solution. We therefore need consider only

$$F(n) - \sum\nolimits_1 F(n - a_i) + \ldots = e(n), \tag{1.2}$$

with n an integer > 0, as before.

From the generating identity

$$\prod_{i=1}^{r} (1 - y^{a_i})^{-1} = \sum_{n=0}^{\infty} A(n) y^n, \quad |y| < 1,$$

we have

$$1 = \prod_{i=1}^{r} (1 - y^{a_i}) \sum_{n=0}^{\infty} A(n) y^n.$$

Hence

$$\sum_{n=1}^{\infty} e(n)y^n = \Big(1 - \sum_1 y^{a_i} + \sum_2 y^{a_i+a_j} - \ldots\Big)\sum_{n=0}^{\infty} A(n)y^n,$$

$$= \sum_{n=0}^{\infty}\Big[A(n) - \sum_1 A(n-a_i) + \ldots\Big]y^n.$$

From the reversibility of the steps, and the uniqueness of the power series expansion of an analytic function (of y, here), the proposition follows.

2. *Determinant for $A(n)$.* Determinant expressions for $A(n)$ have been given by E. Fergola and C. Sardi.* A much simpler form is given at once by (1.2). We denote by $\phi_a(n)$, or simply $\phi(n)$, where $a = a_1 + \ldots + a_r$, the number of partitions of n into an even number of distinct parts chosen from $a_1, \ldots, a_r$, minus the like for an odd number of distinct parts, for $0 \leqslant n \leqslant a$. By convention, $\phi(0) = 1$; and if n does not lie in the interval indicated, $\phi(n) = 0$. Then

$$\prod_{i=1}^{r} (1-y^{a_i}) = \sum_{n=0}^{\infty} \phi(n)y^n;$$

$$\sum_{n=0}^{\infty} e(n)y^n = \sum_{n=0}^{\infty} \phi(n)y^n \sum_{n=0}^{\infty} A(n)y^n;$$

$$e(n) = \phi(0)A(n) + \phi(1)A(n-1) + \ldots + \phi(n)A(0).$$

Replacing n by $n, n-1, \ldots, 0$ in the last, and solving the resulting system of equations, we find, for $n > 0$,

$$A(n) = (-1)^n \begin{vmatrix} \phi(1) & \phi(2) & \phi(3) & .. & \phi(n-1) & \phi(n) \\ 1 & \phi(1) & \phi(2) & .. & \phi(n-2) & \phi(n-1) \\ 0 & 1 & \phi(1) & .. & \phi(n-3) & \phi(n-2) \\ 0 & 0 & 1 & .. & \phi(n-4) & \phi(n-3) \\ & .. & & .. & & .. \\ 0 & 0 & 0 & .. & 1 & \phi(1) \end{vmatrix}$$

As an example, for $r = 2$ and $a_1 = 2$, $a_2 = 3$, $n = 5$, we have $\phi(0) = 1$, $\phi(2) = -1$, $\phi(3) = -1$, $\phi(5) = 1$, and the determinant gives $A(5) = 1$, which is correct, since $2x + 3y = 5$ has only the solution $(x, y) = (1, 1)$. For the same a_1, a_2 we get $A(6) = 2$, correct, since $2x + 3y = 6$ has only the solutions $(x, y) = (3, 0)$, $(0, 2)$.

* *Giornale di Matematiche*, 1 (1863), 63-64; *ibid.* 3 (1865), 94-99, 377-380. See also Dickson's *History*, 2, 125.

3. *Reciprocity.* Consider the two sets of integers $a_1, \ldots, a_r > 0$, $b_1, \ldots, b_s > 0$, and the associated denumerants $A(n)$, $B(n)$. For the first set we have defined $\phi_a(n)$; for the second set we define $\phi_b(n)$ similarly. Between the denumerants A, B there is a simple reciprocity:

$$\begin{vmatrix} A(n) & \phi_b(1) & \phi_b(2) & \ldots & \phi_b(n) \\ A(n-1) & \phi_b(0) & \phi_b(1) & \ldots & \phi_b(n-1) \\ A(n-2) & 0 & \phi_b(0) & \ldots & \phi_b(n-2) \\ & \ldots & \ldots & \ldots & \\ A(0) & 0 & 0 & & \phi_b(0) \end{vmatrix}$$

is equal to the determinant obtained from this when A is replaced by B, and b by a.

To prove this, let $r+s=t$, and let $c_1, \ldots, c_t$ be the set composed of $a_1, \ldots, a_r$, $b_1, \ldots, b_s$ in any order. Let $C(n)$ be the denumerant for $c_1, \ldots, c_t$, and $\phi_c(n)$ the function defined similarly to $\phi_a(n)$, $\phi_b(n)$. Then, for $|y| < 1$,

$$\prod_{i=1}^{r} (1-y^{a_i})^{-1} = \sum_{n=0}^{\infty} A(n) y^n, \quad \prod_{i=1}^{s} (1-y^{b_i})^{-1} = \sum_{n=0}^{\infty} B(n) y^n,$$

$$\prod_{i=1}^{t} (1-y^{c_i})^{-1} = \prod_{i=1}^{r} (1-y^{a_i})^{-1} \cdot \prod_{i=1}^{s} (1-y^{b_i})^{-1} = \sum_{n=0}^{\infty} C(n) y^n.$$

Multiplying the last by $\prod_{i=1}^{s} (1-y^{b_i})$, also by $\prod_{i=1}^{r} (1-y^{a_i})$, and equating coefficients of y^n in the results, we get

$$C(n)+\phi_a(1)C(n-1)+\ldots+\phi_a(n)C(0)=B(n),$$
$$C(n)+\phi_b(1)C(n-1)+\ldots+\phi_b(n)C(0)=A(n).$$

Replacing n by $n, n-1, \ldots, 0$ in each of these and solving the resulting systems for $C(n)$ we get two determinants for $C(n)$. The necessary equality of these is the reciprocity in question.

4. *Certain partition sums.* Let Σ refer to all sets of integer solutions $(i_1, i_2, \ldots, i_n)$ of

$$i_1+2i_2+3i_3+\ldots+n\,i_n=n,$$

$i_1 \geqslant 0, \ldots, i_n \geqslant 0$. Jacobi* gave a purely combinatorial proof of

$$\sum \frac{1}{1^{i_1}2^{i_2}\ldots n^{i_n}.\ i_1!\,i_2!\ldots i_n!} = 1, \tag{4.1}$$

* *Journal für Mathematik*, 22 (1841), 372-4. The same result was stated by several later writers, including Sylvester (1861) and Catalan (1886). Cauchy (*Oeuvres*, t. VI, p. 113), in 1841, gave similar results.

which seems to have been first noted as a consequence of equating coefficients in the identity

$$(1-y)^{-1} = e^{-\log(1-y)} = e^{y}e^{y^2/2}e^{y^3/3}\ldots, \quad |y| < 1.$$

Without attempting combinatorial proofs we note a more general result, (4.4) below, of a similar kind. For y, x suitably restricted to secure absolute convergence, we have

$$(1-y)^{-x} = e^{xy/1}\, e^{xy^2/2}\, e^{xy^3/3}\ldots,$$

and hence, on comparing coefficients of y^n,

$$\sum \frac{x^{i_1+i_2+\ldots+i_n}}{1^{i_1}.2^{i_2}\ldots n^{i_n}.\, i_1!\, i_2!\ldots i_n!} = x(x+1)\ldots(x+n-1)/n!. \quad (4.2)$$

For $x=1$, this gives (4.1). If in the derivative of (4.2) with respect to x we put $x=1$, we get

$$\sum \frac{i_1+\ldots+i_n}{1^{i_1}\ldots n^{i_n}.\, i_1!\ldots i_n!} = 1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\tfrac{1}{n}. \quad (4.3)$$

Let (n, r) denote the sum of all products r at a time of distinct integers chosen from $1, 2, \ldots, n$, and let $(n, 0)=1$. Then

$$(x+1)(x+2)\ldots(x+n) = x^n+(n, 1)x^{n-1}+\ldots+(n, n).$$

If $P(x, y)$ is any polynomial in x, y, we have

$$\sum \frac{P(i_1+\ldots+i_n, y)}{1^{i_1}.2^{i_2}\ldots n^{i_n}.\, i_1!\, i_2!\ldots i_n!} = \frac{1}{n!}\sum_{t=0}^{n-1}(n-1, t)P(n-t, y). \quad (4.4)$$

This follows from (4.2). Replace x by e^x in (4.2) and equate coefficients of x^s in the expanded result. Then, if s is an integer $\geqslant 0$,

$$\sum \frac{(i_1+\ldots+i_n)^s}{1^{i_1}\ldots n^{i_n}.\, i_1!\ldots i_n!} = \frac{1}{n!}\sum_{t=0}^{n-1}(n-1, t)(n-t)^s. \quad (4.5)$$

If now $P(x, y) = p_0x^h+p_1x^{h-1}y+\ldots$, we replace s by h in (4.5) and multiply by p_0, then replace s by $h-1$ and multiply by p_1y, and so on. Adding all the results we find (4.4). Conversely, for $y=0$, $h=s$, (4.4) gives (4.5). For $s=0$, (4.5) gives (4.1). Thus (4.4) includes all the results in this section.

5. *Connection with recurring series.* By successive applications of the operator xd/dx to $x/(1-x)$ we sum the series

$\sum_{r=1}^{\infty} r^s x^r$. Thus for $s = 4$,

$$\sum_{r=0}^{\infty} r^4 x^r = \frac{x(1+11x+11x^2+x^3)}{(1-x)^5}, \ |x| < 1. \qquad (5.1)$$

Generally (s an integer > 0), the sum is a rational function of the form $xQ_s(x)/(1-x)^{s+1}$, where $Q_s(x)$ is a reciprocal polynomial in x of degree $s-1$ with integer coefficients, the coefficients of the highest and lowest powers of x being each 1. Resolving $Q_4(x)$ into its linear factors, we have

$$1+11x+11x^2+x^3 = (1+x)(1+\alpha x)(1+\beta x);$$

here $\alpha = 5+2\sqrt{6}$, $\beta = 5-2\sqrt{6}$, but the actual form of the roots of $Q_s(x) = 0$ in any case is not needed. Hence

$$\sum_{r=1}^{\infty} r^4 x^r = e^{\log(1+x)+\log(1+\alpha x)+\log(1+\beta x)-5\log(1-x)}$$

Expanding the logs in the exponent, we get

$$\sum_{n=1}^{\infty} \frac{x^n}{n} [5-(-1)^n+(-1)^{n-1}(\alpha^n+\beta^n)].$$

It is easily seen that the expression in square brackets is the solution w_n of the difference equation

$$w_{n+2}+10w_{n+1}+w_n = 4[15+2(-1)^n],$$

which is defined by the initial values $w_0 = 2$, $w_1 = 16$. With $\sum$ as in § **4**, we have

$$\sum \frac{w_1^{i_1} w_2^{i_2} \ldots w_n^{i_n}}{1^{i_1} 2^{i_2} \ldots n^{i_n} . \, i_1! \, i_2! \ldots i_n!} = (n+1)^4.$$

In the same way, if $v_{n+2} - pv_{n+1} + qv_n = 0$, $v_0 = 2$, $v_1 = p$, we find

$$\sum \frac{(-1)^{i_1+i_2+\ldots+i_n} v_1^{i_1} v_2^{i_2} \ldots v_n^{i_n}}{1^{i_1} . 2^{i_2} . \ldots n^{i_n} . \, i_1! \, i_2! \ldots i_n!} = 0, \ n \geqslant 2.$$

AN INVERSION FORMULA

By BRIJ MOHAN *and* R. V. SHASTRY, *Benares University.*

[Received 10 December 1937]

1. Bateman* has proved that the inversion formula

$$\int_s^\infty \int_{-\infty}^\infty J_0(x-s)\frac{J_1(x-t)}{x-t} f(t)\,dt\,dx = 2f(s)$$

holds for a certain class of functions $f(x)$. This formula has been generalized by Hardy† who has obtained one containing Bessel functions of orders ν and $1-\nu$. Fox‡ has obtained the following result on somewhat different lines:

THEOREM A. *If $f(s)$ be defined by either of the formulae*

$$f(s) = \int_a^b \phi(\mu) \begin{matrix}\cos \\ \sin\end{matrix} \mu s\, d\mu,$$

where

$$\int_a^b |\phi(\mu)|\, d\mu$$

exists, and $-1<a<b$, then

$$\int_s^\infty \int_{-\infty}^\infty \left[\left\{(\nu-1)\frac{J_{\nu-1}(x-t)}{x-t} J_\nu(x-s) + \nu \frac{J_\nu(x-t)}{x-t} J_{\nu-1}(x-s)\right\} \times f(t)\,dt\right] dx = 2f(s), \qquad (1)$$

where ν is an integer or zero.

1.1. Fox has further shown in the same paper, that (1) holds for functions satisfying the relation

$$f(s) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{\sin(s-t)}{s-t} f(t)\,dt.$$

1.2. Brij Mohan§ has further generalized the formula in the following

THEOREM B. *If $f(s)$ be defined by either of the formulae*

$$f(s) = \int_a^b \phi(\mu) \begin{matrix}\cos \\ \sin\end{matrix} \mu s\, d\mu,$$

* The inversion of a definite Integral, *Proc. Lond. Math. Soc.* (2), 4 (1906), 484.

† An integral equation, *Ibid.* (2), 7 (1909), 445-72.

‡ A generalization of an integral equation due to Bateman, *Ibid.* (2), 26 (1927), 395-402.

§ Some inversion formulae, *Proc. Benares Math. Soc.* 15 (1933), 1-20.

or
$$f(s)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(s-t)}{s-t}f(t)\,dt,$$
then
$$\int_s^{\infty}dx\int_{-\infty}^{\infty}\{J_{\lambda+1}(x-s)J_{\lambda-1}(x-t)+J_\lambda(x-s)J_\lambda(x-t)$$
$$+J_\nu(x-s)J_\nu(x-t)+J_{\nu-1}(x-s)J_{\nu+1}(x-t)\}f(t)dt$$
$$=4f(s). \qquad (2)$$

2. The object of this note is to show that the results (1) and (2) are true for another class of functions $f(x)$.

THEOREM 1. *If $f(x)$ be defined by*
$$f(x)=\int_a^b\frac{\sin(w-x)}{w-x}\phi(w)\,dw, \qquad (3)$$
where
$$\int_a^b|\phi(\mu)|\,d\mu$$
exists, then (1) *and* (2) *are true.*

PROOF: Fox has proved the result
$$\int_s^{\infty}\left\{(\nu-1)\frac{J_\nu(x-t)}{x-t}J_\nu(x-s)+\nu\frac{J_\nu(x-t)}{x-t}J_\nu(x-s)\right\}dx$$
$$=\frac{2}{\pi}\frac{\sin(s-t)}{s-t}. \qquad (4)$$
Multiply both the sides of (4) by $f(t)$ and integrate from $-\infty$ to ∞.

Then,
$$\int_{-\infty}^{\infty}f(t)\,dt\int_s^{\infty}\left\{(\nu-1)\frac{J_\nu(x-t)}{x-t}J_\nu(x-s)+\nu\frac{J_\nu(x-t)}{x-t}J_{\nu-1}(x-s)\right\}dx$$
$$=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin(s-t)}{s-t}f(t)\,dt$$
$$=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin(s-t)}{s-t}dt\int_a^b\frac{\sin(w-t)}{w-t}\phi(w)dw. \qquad (5)$$
making use of (3).

Now, changing the order of integration in the repeated integral on the right side of (5), we get
$$\int_{-\infty}^{\infty}f(t)dt\int_s^{\infty}\left\{(\nu-1)\frac{J_{\nu-1}(x-t)}{x-t}J_\nu(x-s)+\nu\frac{J_\nu(x-t)}{x-t}J_{\nu-1}(x-s)\right\}dx$$
$$=\frac{2}{\pi}\int_a^b\phi(w)dw\int_s^{\infty}\frac{\sin(s-t)}{s-t}\,\frac{\sin(w-t)}{w-t}\,dt, \qquad (6)$$
the inversion being valid since both the integrals are absolutely convergent

Now

$$\int_{-\infty}^{\infty} \frac{\sin(s-t)}{s-t}\,\frac{\sin(w-t)}{w-t}\,dt$$

$$=\frac{1}{s-w}\int_{-\infty}^{\infty} \frac{\sin(s-t)\sin(w-t)}{w-t}\,dt-\frac{1}{s-w}\int_{-\infty}^{\infty} \frac{\sin(s-t)\sin(w-t)}{s-t}\,dt.$$

Putting $w-t=y$ in the first and $s-t=z$ in the second, we have

$$\int_{-\infty}^{\infty} \frac{\sin(s-t)}{s-t}\,\frac{\sin(w-t)}{w-t}\,dt=\frac{1}{s-w}\int_{-\infty}^{\infty}\frac{\sin y}{y}\sin(y+s-w)\,dy$$

$$+\frac{1}{s-w}\int_{-\infty}^{\infty}\frac{\sin z}{z}\,\sin(z-s+w)\,dz$$

$$=\pi\frac{\sin(w-s)}{w-s}$$

Hence (6) becomes, in virtue of (3)

$$\int_{-\infty}^{\infty} f(t)\,dt\int_{s}^{\infty}\left\{(\nu-1)\frac{J_\nu(x-t)}{x-t}J_\nu(x-s)+\nu\frac{J_\nu(x-t)}{x-t}J_{\nu-1}(x-s)\right\}dx$$

$$=2f(s). \tag{7}$$

Hence by changing the order of integration on the left side of (7), we arrive at Fox's result.

2.1. Similarly we can prove (2) by starting with the result

$$\int_{s}^{\infty}\Big\{J_{\lambda+1}(x-s)J_{\lambda-1}(x-t)+J_\lambda(x-s)J_\lambda(x-t)+J_\nu(x-s)J_\nu(x-t)$$

$$+J_{\nu-1}(x-s)J_{\nu+1}(x-t)\Big\}dx$$

$$=\frac{4}{\pi}\,\frac{\sin(s-t)}{s-t}.$$

2.2. THEOREM 2. *If* $f(x)$ *be defined by*

$$f(x)=\int_{a}^{b}\frac{\sin\mu(w-x)}{w-x}\phi(w)\,dw,$$

where

$$\int_{a}^{b}|\phi(w)|\,dw$$

exists and $0<\mu\leqslant 1$, *then* (1) *and* (2) *are true.*

To prove this, we make use of Hardy's* formula

$$\int_{-\infty}^{\infty}\frac{\sin m(t-x)}{t-x}\,\frac{\sin\mu(t-w)}{t-w}\,dt=\pi\frac{\sin\mu(w-x)}{w-x}\,,\quad 0<\mu\leqslant m,$$

put $m=1$, and proceed as before.

* *Loc. cit.*

CANONICAL BASIS FOR IDEALS IN A POLYNOMIAL DOMAIN OVER A COMMUTATIVE RING WITH FINITE BASIS FOR IDEALS.

By R. VAIDYANATHASWAMY *and* T. VENKATARAYUDU.

[Received 16 February 1938]

1. Let A be a commutative ring with unit element and $A[x]$ denote the ring of polynomials in x with coefficients from A. Hilbert's theorem is:—

If every ideal in A *has a finite basis, then every ideal in* $A[x]$ *also has a finite basis.*

The proof of this as given by Van der Waerden in his *Moderne Algebra* Bd. II gives us no information beyond the fact that every ideal in $A[x]$ has a finite basis. If we just reverse the argument in his proof we will be able to give actually a canonical basis for every ideal in $A[x]$ which will be found to be a powerful tool in several applications.

We assume the theorem that in a ring in which every ideal has a finite basis, a chain of ideals in which every ideal is a divisor of the previous one must be finite and conversely.

We prove now our principal

THEOREM A.—*Every ideal I in* $A[x]$ *has a basis in the form*

$$(a_{11}, a_{12}, \ldots, a_{1r_1}, a_{21}, a_{22}, \ldots, a_{2r_2}, \ldots, a_{t1}, a_{t2}, \ldots, a_{tr_t})$$

where $a_{k1}, a_{k2}, \ldots, a_{kr_k}$ *are polynomials in* $A[x]$ *whose highest coefficients form an ideal* b_k *which is a true divisor of* b_{k-1}.

PROOF:—Let n_1 be the least degree of the polynomials in I, (n_1 may be zero). The highest coefficients of the polynomials of degree n_1 together with zero form an ideal in A, for if α and β are the highest coefficients of the polynomials a, b of degree n, $a-b$ is a polynomial of degree n_1, or is identically zero, and the highest co-

efficient of $a-b$ is $\alpha-\beta$ or 0. Further if α is the highest coefficient of a, $\lambda\alpha$ is the highest coefficient of the polynomial λa. Hence the highest coefficients of the polynomials of degree n_1 together with zero form an ideal, say b_1, which by hypothesis has a basis, say $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1r_1}$. Let $a_{11}, a_{12}, \ldots, a_{1r_1}$ be a set of polynomials in I whose highest coefficients are respectively $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1r_1}$. We take $a_{11}, a_{12}, \ldots, a_{1r_1}$, as some basis elements in I. It follows from above that every polynomial in I of degree n_1 can be expressed in terms of $a_{11}, a_{12}, \ldots, a_{1r_1}$. For let

$$f = a_0 x^{n_1} + a_1 x^{n_1 - 1} + \ldots + a_{n_1}$$

be any polynomial in I of degree n_1. If $a_0=0$, f itself is zero. Therefore $a_0 \neq 0$. a_0 is an element of the ideal $(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1r_1})$; let $a_0 = \lambda_1\alpha_{11} + \lambda_2\alpha_{12} + \ldots + \lambda_{r_1}\alpha_{1r_1}$.

The polynomial $\phi = \lambda_1 a_{11} + \lambda_2 a_{12} + \ldots + \lambda_{r1} a_{1r_1}$ is of degree n_1 and the highest co-efficient is a_0. Hence $f-\phi$ is a polynomial of degree $< n_1$ and is hence identically zero.

Hence $f = \lambda_1 a_{11} + \lambda_2 a_{12} + \ldots + \lambda_{r_1} a_{1r_1}$, and therefore every polynomial in I of degree n_1 can be expressed in terms of $a_{11}, a_{12}, \ldots, a_{1r_1}$.

Now polynomials of degree n $(n > n_1)$ occur in I. The highest coefficients of polynomials in I of degree $\leqslant n$ form an ideal C in A. Since all elements of b_1 are elements in C, C is a divisor of b_1. If all the elements of C are elements in b_1, $C = b_1$. On the other hand if there is at least one element in C which does not belong to b_1, we say that C is a true divisor of b_1. Let n_2 be the least n for which the ideal formed by the highest coefficients of polynomials of degree $\leqslant n_2$ form an ideal b_2 which is a true divisor of b_1. By hypothesis b_2 has a basis, say, $(\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2r_2})$. Let $a_{21}, a_{22}, \ldots, a_{2r_2}$ be a set of polynomials whose highest coefficients are $\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2r_2}$ respectively. We show in the first place that polynomials in I of degree $< n_2$ can be expressed in terms of $a_{11}, a_{12}, \ldots, a_{1r_1}$. Let $f = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$ be a polynomial whose degree $n < n_2$. a_0 is therefore an element of $(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1r_1})$. Let $a_0 = \lambda_1\alpha_{11} + \lambda_2\alpha_{12} + \ldots + \lambda_{r_1}\alpha_{1r_1}$. Take the polynomial $\phi_1 = \lambda_1 a_{11} + \ldots + \lambda_r \alpha_{1r_1}$. $f-\phi_1$ is a polynomial in I whose degree is $\leqslant n-1$ and continuing this process we find $f-\phi_1-\phi_2\ldots-\phi_{n-n_1}$ is a polynomial in I of degree n_1 which, as we have already shown, can be expressed as a polynomial ϕ_{n-n_1-1} in terms of $a_{11}, \ldots, a_{1r_1}$. Hence $f = \phi_1 + \phi_2 + \ldots + \phi_{n-n_1} + \phi_{n-n_1-1}$.

We can show as in the previous case that any polynomial in I of degree n_2 can be expressed in terms of $a_{21}, a_{22},\ldots a_{2r_2}$ and $a_{11}, a_{12},\ldots, a_{1r_1}$, for if

$$f = a_0x^{n_2}+a_1x^{n_2-1}+\ldots+a_{n_2}$$

is a polynomial in I of degree n_2, a_0 is an element of the ideal $(\alpha_{21}, \alpha_{22},\ldots, \alpha_{2r_2})$ and let

$$a_0 = \mu_1\alpha_{21}+\mu_2\alpha_{22}+\ldots+\mu_{r_2}\alpha_{2r_2}.$$

Take the polynomial $\psi = \mu_1a_{21}+\mu_2a_{22}+\ldots+\mu_{r_2}a_{2r_2}$. It is of degree n_2 and the highest coefficient is a_0. and the polynomial $f-\psi$ is in I and of degree $<n_2$, and therefore can be expressed in terms of $a_{11}, a_{12},\ldots a_{1r_1}$.

In continuing the construction as indicated above we get a chain of ideals $b_1, b_2,\ldots, b_{k-1}, b_k,\ldots$ in which each is a true divisor of the previous one and therefore the chain must be of finite length, say, let it stop with b_t. Let the basis of b_t be $(\alpha_{t_1}, \alpha_{t_2},\ldots, \alpha_{tr_t})$ and let $a_{t1}, a_{t2},\ldots, a_{tr_t}$ be a set of polynomials in I whose highest coefficients are $\alpha_{t_1}, \alpha_{t2},\ldots, \alpha_{tr_t}$ respectively. We prove as above that every element in I can be expressed in terms of $(a_{11}, a_{12},\ldots, a_{1r_1}, a_{21}, a_{22},\ldots, a_{2r_2},\ldots, a_{t1}, a_{t2},\ldots, a_{tr_t})$.

2. *Ideal theory of the integral polynomials.*

Since every ideal in C is a principal ideal the canonical basis for ideals in $C[x]$ is simple. We have

THEOREM B. *Every ideal* I *in* $C[x]$ *can be put into the canonical form*

$$(Mx^{n_1}, M_1x^{n_1} f_1(x), M_2x^{n_1} f_1(x)f_2(x),\ldots),$$

where M_r *is a factor of* M_{r-1}.

PROOF:—Let n_1 be the least degree of the polynomials in I. The highest coefficients of the polynomials in I of degree n_1 together with zero form an ideal say (M). Let a_1 be a polynomial in I whose highest coefficient is M. Now $a_1-Mx^{n_1}=0$ and hence $a_1 = Mx^{n_1}$. Let n_2 be the least degree of the polynomials in I such that the highest coefficients of the polynomials in I of degree $\leqslant n_2$ form an ideal (M_1) which is a true divisor of M. Let $\phi(x)$ be the polynomial of degree n_2 with highest coefficient M_1. If $M_1N_1=M$, $N_1\phi(x)-Mx^{n_2}$ is a polynomial in I of degree $<n_2$ and hence $N_1\phi(x)-Mx^{n_2}=A_1Mx^{n_1}$ where A_1 is a polynomial in $C[x]$ and hence $\phi(x)$ is of the form $M_1x^{n_1}f_1(x)$. Similarly let n_3 be the least degree of the polynomials in I such that the highest coefficients of the polynomials of degree $\leqslant n_3$ form an ideal (M_2)

which is a true divisor of M_1, and let $\psi(x)$ be a polynomial in I whose highest coefficient is M_2. Let $M_2N_2=M_1$. Now the polynomial

$$N_2\psi(x)-M_1x^{n_1}f(x)\ x^{n_3-n_2}$$

is a polynomial of degree $<n_3$ and therefore

$$N_2\psi(x)-M_1x^{n_1}f_1(x)x^{n_3-\lambda 2}=A_1Mx^{n_1}+A_2M_1x_1^nf_1(x),$$

where A_1 and A_2 are two polynomials in $C[x]$. Hence $N_2\ \psi(x)\equiv M_1\ x^{n_1}\ f_1\ (x).\ x^{n_3-n_2}+A_2\ M_1\ x^{n_1}\ f_1(x)$ mod (Mx^{n_1}) and $\psi(x)$ is of the form $M_2x^{n_1}\ .\ f_1(x)\ f_2(x)\ (\text{mod}\ Mx^{n_1})$.

The process breaks after a finite number of steps and the theorem is established.

THEOREM 1. *Every prime ideal in* $C[x]$ *has one of the three forms:*

(1) $P=\Big(p, f\ (x)\Big)$, *p a prime, $f(x)$ an irreducible polynomial.*

(2) $P=(p)$

and (3) $P=\Big(f\ (x)\Big)$.

THEOREM 2. *Every primary ideal in* $C[x]$ *has one of the three forms:*

$$(1)\ q=\Big(p^{\sigma_1}, p^{\sigma_2}.\ f_1^{\rho_1}(x), p^{\sigma_3}.\ f_1^{\rho_2}\ (x),\ldots, f_1^{\rho_t}(x)\Big),$$

where p is a prime and $f(x)$ is an irreducible polynomial

$$\sigma_1>\sigma_2>\sigma_3\ldots,\rho_1<\rho_2<\ldots$$

(2) $q=(p^{\sigma})$

and (3) $q=\Big(f^{\rho}(x)\Big)$.

(1) requires proof, the other cases are evident. In the first instance q is a primary ideal for $\Big(p, f(x)\Big)^{\lambda}\equiv o(q)\equiv o\Big(p, f(x)\Big)$ $\lambda>\sigma_1, \rho_t$ and $\Big(p, f(x)\Big)$ is a factorless ideal. Hence q is primary.

On the other hand every primary ideal should be of this form. For, in constructing the canonical basis for it, it is obvious that the first term should be a power of a prime. If the second term is $p^{\sigma_2}.f_1^{\rho_1}(x).f_2^{\rho_2}(x)$ the subsequent basis elements must be multiples

of $f_1^{\rho_1}(x).f_2^{\rho_2}(x)$. Now $p^{\sigma_2}.f_1^{\rho_1}(x)\,f_2^{\rho_2}(x) \equiv o(q)$. But $p^{\sigma_2}f_2^{\rho_2}(x) \not\equiv o(q)$. Hence $f_1^{\rho_1}(x) \equiv o(q)$ for sufficiently great ρ and hence

$$f_1^{\rho}(x) = p^{\sigma_1} + A_1 p^{\sigma_2}\, f_1^{\rho_1}(x).f_2^{\rho_2}(x) + \text{multiples of } f_1^{\rho_1}(x) f_2^{\rho_2}(x) \tag{1}$$

which is absurd, for whenever there is $f_1(x)$ there is also $f_2(x)$ with it. A relation of type (1) can therefore arise only when $A_1, A_2, \ldots$ are all zero and $f_1^{\rho}(x)$ is not a multiple of p^{σ_1}. Hence the second term should be $p^{\sigma_2}.f_1^{\rho_1}(x)$. For a similar reason, the third should be of the form $p^{\sigma_3}.f_1^{\rho_2}(x)$, and the last term should be a power of the irreducible polynomial $f_1(x)$, otherwise

$$f^{\rho}(x) = A_1 p^{\sigma_1} + A_2 p^{\sigma_2}\, f_1^{\rho_1}(x) + \ldots \tag{2}$$

and the right hand side is a multiple of p and therefore for a relation of type (2) there must be a term which is independent of p.

ON LAGUERRE POLYNOMIALS WHICH ARE SELF-RECIPROCAL IN THE HANKEL TRANSFORM

By R. S. VARMA, *Cawnpore.*

[Received 7 February 1938]

In a recent paper*, S. C. Mitra has shown that

$$\int_0^\infty z^{n-\frac{1}{2}} e^{-\frac{1}{4}z^2} D_{2n}(z) J_{n-1}(xz)\,dz$$
$$= (-)^n x^{n-\frac{1}{2}} e^{-\frac{1}{4}x^2} D_{2n}(x)$$

and

$$\int_0^\infty z^{n-\frac{1}{2}} e^{-\frac{1}{4}z^2} D_{2n-1}(z) J_n(xz)\,dz$$
$$= (-)^{n-1} x^{n-\frac{1}{2}} e^{-\frac{1}{4}x^2} D_{2n-1}(x).$$

More recently† I have shown that these results are particular cases of the general integral

$$\int_0^\infty y^{2l-n-1} e^{-\frac{1}{4}y^2} J_n(xy) W_{k,m}(\tfrac{1}{2}y^2)\,dy$$
$$= \frac{2^{l-n-1} x^n\, \Gamma(l+m+\frac{1}{2})\, \Gamma(l-m+\frac{1}{2})}{\Gamma(n+1)\, \Gamma(l-k+1)}\, {}_2F_2\left[\begin{matrix} l+m+\frac{1}{2},\ l-m+\frac{1}{2} \\ n+1,\ l-k+1 \end{matrix}; -\tfrac{1}{2}x^2\right]$$
$$l \pm m + \tfrac{1}{2} > 0. \qquad (1)$$

I now find that the relation (1) gives the following general

THEOREM. *The function*

$$\phi(x) = x^{2m+\mu+\frac{1}{2}} e^{-\frac{1}{2}x^2} L_\mu^m(\tfrac{1}{2}x^2)$$
$$\mu + 2m + 1 > 0$$

is self-reciprocal or skew self-reciprocal in the Hankel transform of order $\mu+2m$ *according as* μ *is even or odd.*

To prove this, we obviously see that when

$$l+m+\tfrac{1}{2} = n+1 \text{ or } l = n-m+\tfrac{1}{2}$$

* S. C. Mitra, 'On parabolic cylinder functions which are self-reciprocal in the Hankel transform', *Math. Zeit.*, 43 (1937), 205-11.

† R. S. Varma, 'Some infinite integrals involving Weber's parabolic cylinder functions', *Journ. Indian Math. Soc.* (2) 3 (1938), 25-33.

the generalised hypergeometric function on the right of (1) reduces to Kummer's function. Now since

$$L_n^m(x) = \frac{\Gamma(m+n+1)}{n!\,\Gamma(m+1)}\,{}_1F_1(-n\,;\,m+1\,;\,x)$$

$$= \frac{(-)^n}{n!}\,x^{-\frac{1}{2}(m+1)}e^{\frac{1}{2}x}W_{n+\frac{1}{2}m+\frac{1}{2},\,\frac{1}{2}m}(x)$$

we get from (1)

$$\int_0^\infty y^{n+1}e^{-\frac{1}{2}y^2}J_n(xy)L_\mu^{2m}(\tfrac{1}{2}y^2)\,dy$$

$$= (-)^\mu x^n e^{-\frac{1}{2}x^2}L_\mu^{n-\mu-2m}(\tfrac{1}{2}x^2). \qquad (2)$$

If we set $n=\mu+4m$ in (2), we obtain finally

$$\int_0^\infty \sqrt{xy}\,J_{\mu+2m}(xy)\,y^{2m+\mu+\frac{1}{2}}e^{-\frac{1}{2}y^2}L_\mu^m(\tfrac{1}{2}y^2)\,dy$$

$$= (-)^\mu x^{2m+\mu+\frac{1}{2}}e^{-\frac{1}{2}x^2}L_\mu^m(\tfrac{1}{2}x^2),$$

which is equivalent to the theorem stated above.

The following three special cases of the theorem are rather interesting:

Case (*i*) For $m=-\frac{1}{2}$ and $m=\frac{1}{2}$, we get Mitra's result given above.

Case (*ii*) For $m=0$, we obtain that the function

$$\phi(x) = x^{\mu+\frac{1}{2}}e^{-\frac{1}{2}x^2}L_\mu(\tfrac{1}{2}x^2)$$

is self-reciprocal or skew self-reciprocal in the Hankel transform of order μ according as μ is even or odd.

Case (*iii*) For $m=1$, we obtain that the function

$$\phi(x) = x^{\mu+\frac{1}{2}}e^{-\frac{1}{4}x^2}k_{2\mu+2}\left(\frac{x^2}{4}\right)$$

is self-reciprocal or skew self-reciprocal in the Hankel transform of order $\mu+2$ according as μ is even or odd.

*ON THREE RATIONAL FUNCTIONS ON A RIEMANN SURFACE.

By K. S. SURYANARAYANAN, *Madras University.*

I

1. *Introductory.*

Let y be an algebraic function of z defined by the irreducible relation

$$F(y, z) \equiv A_0(z)y^n + A_1(z)y^{n-1} + \ldots + A_n(z) = 0 \qquad (1)$$

where the A's are polynomials in z of degree, say, m. The manifold of pairs of values y, z which satisfy the given equation (1) constitute the *Riemann surface* of the algebraic function y, constructed in the usual manner. Any rational function of y and z is a rational function of position on this Riemann surface. The *order* of such a rational function is the number of places on the surface at which it takes an assigned value. In particular, y and z are rational functions of position on the surface of orders m and n respectively.

The totality of the rational functions on the Riemann surface defined by (1) forms a corpus. This corpus is the algebraic extension of the corpus $K(z)$ of rational functions of z by the adjunction of the algebraic function y of z to $K(z)$. y is called a *primitive* or *generating* element of the extended corpus inasmuchas every element of that corpus can be expressed as a rational function of y and z. If, for example, η be any other element of the corpus satisfying the irreducible relation $F(\eta, z) = 0$, where $F(\eta, z)$ is of degree n in η, η is also a primitive element of the corpus, or in other words, any element of the corpus is also a rational function of η and z so that there is a birational correspondence between $F(y, z) = 0$ and $F(\eta, z) = 0$.

Let us now suppose x, y, z to be three rational functions of orders l, m, n respectively on an assigned Riemann surface R. We can evidently give z† the role of the independent variable and

* I thank Dr. Vaidyanathaswamy for help in the preparation of this paper.

† Heinrich W. E. Jung, *Einführung in die Theorie der Algebraischen Funktionen einer Veranderlichen*, p. 66.

generate the corpus of rational functions on the Riemann surface as an algebraic extension of the corpus $K(z)$ of rational functions of z. (Geometrically this is equivalent to replacing the surface R by a birationally and topologically equivalent surface with n sheets, n being the order of the rational function z). Suppose x and y are primitive elements of this corpus so that the irreducible relations

$$F_1(x, z)=0, \ F_2(y, z)=0$$

connecting x, y and z are of degrees n in x and l in z, n in y and m in z respectively. Let $\phi(x, y)$ be the z-resultant of F_1 and F_2, which is then of degrees mn in x and ln in y. The present paper is concerned with the reducibility of $\phi(x, y)$.

In geometry, we know the case of Poncelet's theorem* wherein $\phi(x, y)$ splits into two factors, (two being the order of z), each of degree two in x and y where x and y themselves are of order two. The purpose of the present paper is to bring out the corpus-theoretic principle which is the explanation of the factorisability of ϕ. The general result reached is as follows:

Corresponding to every relative automorphism of the corpus of rational functions on the surface (i.e., automorphism which keeps the independent variable z fixed), there is a factor $\psi(x, y)$ of $\phi(x, y)$ of degrees l, m in x, y, which separates out from ϕ.

The Riemann surface with z as the independent variable is n-sheeted. Each relative automorphism of the corpus of rational functions on this surface, therefore, is equivalent to a birational transformation of this surface into itself of such a sort that a point corresponds in this transformation to a certain one of the remaining $n-1$ points on the surface just above or below it. These special birational transformations of the surface into itself are called *deck-transformations.*

In II, I summarise the known theory of the relative automorphisms of any algebraic extension of a fundamental corpus and apply the result to the corpus of rational functions on a Riemann surface, considered as an algebraic extension of $K(z)$. In III, I prove the main result of the paper and in IV, I consider some applications.

* R. Vaidyanathaswamy, A Postscript to Poncelet's Theorem. *Proc. L. M. S.* (2), 27.

II

2. *Relative automorphisms of an algebraic extension of a fundamental corpus.*

Let Δ_1 and Δ_2 be any two algebraic extensions of a fundamental corpus Δ. Δ_1 and Δ_2 are said to be 'isomorphic relatively to Δ' or briefly in *relative isomorphism*, when there is a (1, 1) correspondence between the elements of Δ_1 and Δ_2, preserving addition and multiplication, which leaves the elements of Δ unaltered. If further Δ_1 is identical with Δ_2, the relative isomorphism becomes a relative automorphism.

Let Δ_1 and Δ_2 be algebraic extensions of a fundamental corpus Δ, with a relative isomorphism Σ. In this paper, we assume that all algebraic extensions admit primitive elements.* Then Δ_1 is an algebraic extension of Δ by the adjunction of α to it, so that $\Delta_1 = \Delta(\alpha)$, where α satisfies an irreducible equation $f(\alpha) = 0$, $f(\alpha)$ having its coefficients in Δ. α, then, is a primitive element of Δ_1, and if β in Δ_2 corresponds to α in the relative isomorphism Σ, β should also be a primitive element of Δ_2, and β too satisfies the algebraic relation $f(\beta) = 0$ as the co-efficients of $f(\beta)$ are unaltered in the isomorphism. Thus Δ_1 and Δ_2 arise by adjunction of conjugate algebraic elements to the fundamental corpus and therefore are themselves conjugate algebraic extensions of the fundamental corpus. It is also evident that Σ carries every element of Δ_1 into its conjugate element in Δ_2. Conversely, if all the conjugate algebraic extensions of the fundamental corpus corresponding to the roots of $f(x) = 0$ are distinct, there can be only one relative isomorphism between Δ_1 and Δ_2, namely, that in which α corresponds to β, since, by hypothesis, β is the only primitive element in Δ_2 conjugate to α.

The polynomial $f(x)$ of degree n, defines n algebraic extensions of the fundamental corpus Δ and is irreducible in Δ. It has, therefore, n distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. But even though the roots are distinct, it may well happen that the algebraic extensions are not all distinct. For this case we have

THEOREM 1. *If exactly r of the conjugate corpuses coalesce with $\Delta(\alpha_1)$, then the n corpuses $\Delta(\alpha_i)$ coalesce in groups of r so that n/r of them are distinct and r is a divisor of n.*

Further, each of the n/r distinct corpuses $\Delta(\alpha_i)$ has a group of r relative automorphisms, and these n/r groups are (1, 1) isomorphic with one another.

* For the condition that the algebraic extension of a corpus may admit primitive elements, see Van der Waerden, *Moderne Algebra*, Band I, p. 120.

PROOF: Suppose $\Delta(\alpha_1) = \Delta(\alpha_2) = \ldots = \Delta(\alpha_r) \neq \Delta(\alpha_i)$ $(i = r+1,\ldots, n)$. By what has been proved, there is a relative isomorphism betwen $\Delta(\alpha_1)$ and $\Delta(\alpha_{r+1})$, in which α_1 corresponds to α_{r+1}. Since $\alpha_2,\ldots, \alpha_r$ are primitive elements of $\Delta(\alpha_1)$ satisfying the equation $f(x)=0$, they must correspond in this isomorphism to primitive elements of $\Delta(\alpha_{r+1})$ also satisfying the equation $f(x) = 0$, that is, to elements which are conjugate to $\alpha_2,\ldots, \alpha_r$. Let the elements to which they correspond be $\alpha_{r+2},\ldots, \alpha_{2r}$. Then $\Delta(\alpha_{r+1})=\Delta(\alpha_{r+2}) = \ldots = \Delta(\alpha_{2r})$. By reversing the above argument we can show that $\Delta(\alpha_{r+1})$ does not coalesce with any other conjugate corpus. By continuing this argument we show finally that the n conjugate corpuses $\Delta(\alpha_i)$ coalesce in groups of r so that n/r of them are distinct and that r is a divisor of n.

Since $\alpha_1, \alpha_2,\ldots, \alpha_r$ are mutually distinct, $\Delta(\alpha_1)$ has r relative automorphisms which respectively carry α_1 into $\alpha_1, \alpha_2,\ldots, \alpha_r$. Since further $\Delta(\alpha_1)\neq\Delta(\alpha_i)$ $(i=r+1,\ldots)$ by hypothesis, it has no other relative automorphisms. These r relative automorphisms must evidently form a group which can be represented as a permutation group of the r symbols $(\alpha_1, \alpha_2,\ldots,\alpha_r)$. Since $\Delta(\alpha_1)$ and $\Delta(\alpha_{r+1})$ are relatively isomorphic, the groups of their relative automorphisms are abstractly identical. We can thus speak of the group of relative automorphisms associated with an irreducible relation $f(x) = 0$, meaning thereby the group of automorphisms of any one of the algebraic extensions of the fundamental corpus by the adjunction of a root of $f(x) = 0$.

3. *Application to the case of the algebraic function of a complex variable.*

We shall now apply the above theory to the case of the algebraic function of a complex variable z. We shall take for the fundamental corpus Δ, the corpus $\Delta(z)$ of all complex rational functions of the complex variable z. The irreducible relation in $\Delta(z)$ corresponding to $f(x) = 0$ is of the form

$$F(x, z)=A_0(z)x^n+A_1(z)x^{n-1}+\ldots+ A_n(z) = 0$$

where the A's are polynomials in z of degree, say, m. The manifold of pairs of values z, x satisfying this equation constitute the Riemann surface constructed in the usual manner; and the surface is n-sheeted if we take z to be the independent variable. A rational function of position on this surface is a rational function in the ordinary sense of x and z. All such rational functions of position on the surface, therefore, form a corpus.

But in order to extend as before the fundamental corpus $\Delta(z)$ by the adjunction of a root of $F(x, z) = 0$, we want the n roots

of this equation, say, $x_1, x_2, \ldots, x_n$ as n distinct entities. But the distinctness of each of these roots has become obliterated in the concept of the Riemann surface. The same rational function x on the Riemann surface takes these values at the n points having the same z-coordinate. But if we cut up all the passages which connect the various sheets so that no point can move from one sheet to another, each sheet contains only one point with a given z-coordinate, in fact the sheets are really z-planes with cuts along the lines joining the branch points of x *qua* function of z. Therefore the rational function x, being a rational function of position, takes only one value for a given z on a sheet, and the value x_i is taken only on the i-th sheet. Thus, the roots have been exhibited as single-valued functions of z on the cut sheets, which cannot pass over into one another. These roots, therefore, may be adjoined to $\Delta(z)$ to get the n algebraic extensions which may or may not all be distinct. By our general result, these n extensions are in relative isomorphism with one another by means of the correspondence $x_i \rightarrow x_j$, $z \rightarrow z$. We now prove the following result:

Each of the n corpuses $\Delta(z, x_i)$ is also in relative isomorphism with the corpus of rational functions on the Riemann surface.

For, the one-valued function of position x takes the same values at points of the i-th sheet of the Riemann surface as the one-valued function x_i at those points of the separated i-th sheet. If we patch up the cuts and extend analytically the value of the function x_i along all possible paths throughout the Riemann surface, x_i takes the same value at points of the Riemann surface as the rational function x does. The one-valued function of position z on the separated i-th sheet takes the same values at points of the restored Riemann surface as the rational function z on the surface. Also any rational function $R(z, x_i)$ of z and x_i, treated in the same way, takes the same values on the uncut surface as the rational function $R(z, x)$. Hence the relative isomorphism between $\Delta(z, x_i)$ and the corpus of rational functions on the Riemann surface follows.

As a consequence of this relative isomorphism, it follows that if the corpuses $\Delta(z, x_i)$ coalesce in groups of r (r, a divisor of n) in accordance with Theorem 1, so that $\Delta(z, x_i)$ admits a group G_r, of order r, of relative automorphisms, then the corpus of rational functions on the Riemann surface also admits the group G_r of relative automorphisms. Since a position on the Riemann surface may be identified with a pair of values of x, z which satisfy the equation $F(x, z) = 0$, a relative automorphism of the corpus of

rational functions on the surface may be defined by $x' \to x$, $z \to z$; $F(x, z) = F(x', z) = 0$. A relative automorphism of the corpus of rational functions on the surface must, therefore, be interpreted as a deck-transformation $a \to g(a)$ of positions on the surface such that the rational function x' corresponding to x, takes the same value at $g(a)$ as the function x takes at a. Thus the Riemann surface admits the group G_r of deck-transformations. In particular if $r = n$, the equation $F(x, z) = 0$ is Galoisian over $\Delta(z)$ and the associated Riemann surface is *regular*,* admitting a group of n deck-transformations so that corresponding to any two points having the same z-coordinate, we have a deck-transformation of the surface carrying one of them into the other.

III

4. *Factorisability of the resultant connected with three rational functions x, y, z.*

If x and y are rational functions of orders m and l on the Riemann surface defined by $F(x, z) = 0$, then to each value of x there are m positions on the surface and therefore m values of y, while to each value of y there are l values of x. Hence there is an algebraic relation $\psi(x, y) = 0$ connecting x, y of degree l in x and m in y. The polynomial ψ may or may not be irreducible.

Hence also there is a relation $F'(y, z) = 0$ connecting y and z, of degree n in y and l in z. We assume this relation to be irreducible, so that y is also a primitive element of the corpus of rational functions on the surface and the relations $F(x, z) = 0$, $F'(y, z) = 0$ can be birationally transformed into each other. Denote the z-resultant of $F(x, z)$, $F'(y, z)$ by $\phi(x, y)$ so that ϕ is of degree nl in x and nm in y.

We now prove

THEOREM 2. *$\psi(x, y)$ is a factor of the eliminant $\phi(x, y)$.*

PROOF: The relation $\psi(x, y) = 0$ connects the values of x, y at the same position on the Riemann surface, while the relation $\phi(x, y) = 0$ (from the meaning of elimination) connects the respective values of x, y at two positions with the same z, i.e. directly above or below each other. Hence $\psi(x, y) = 0$ implies $\phi(x, y) = 0$. Hence $\psi(x, y)$ is a factor of $\phi(x, y)$.

This theorem may be generalised as

THEOREM 3. *To each deck-transformation of the Riemann surface associated with the birationally equivalent irreducible equations $F(x, z) = 0$, $F'(y, z) = 0$, there corresponds a factor*

* Kerekjarto, *Vorlesungen uber Topologie* I, p. 162.

$\psi_1(x, y)$ of degree l in x and m in y of the z resultant $\phi(x, y)$ of F, F'.

The previous theorem is the particular case of this, corresponding to the identical deck-transformation.

PROOF: Suppose we have a deck-transformation $a \rightarrow g(a)$. We have seen that to each such deck-transformation there is a relative automorphism $y \rightarrow y_g$ of rational functions on the surface such that y_g takes the same value at $g(a)$ as y takes at a. In other words, to each point a, there is a point $g(a)$ with the same z such that the value of y at a is the same as the value of y_g at $g(a)$. So the orders of the rational functions y and y_g are the same. The two rational functions x and y_g are connected by the relation $\psi_1(x, y_g) = 0$, (of degree l in x and m in y) the values of x and y_g being taken in the same position, say, $g(a)$. Since y_g takes the same value at $g(a)$ as y at a, $\psi_1(x, y) = 0$ connects the values of the rational functions y, x at $a, g(a)$ respectively. As before, since a and $g(a)$ have the same z-coordinate, $\psi_1(x,y) = 0$ connects the values of x and y at points having the same z-coordinate, and hence $\psi_1(x, y) = 0$ implies $\phi(x, y) = 0$, so that ψ_1 is a factor of ϕ.

5. *Irreducibility of the factors ψ.*

Each relation $\psi_1(x, y_g) = 0$ connects the respective values of the rational functions x, y_g at $g(a)$ on the Riemann surface where $a \rightarrow g(a)$ defines a deck-transformation. Therefore by a known* result in the theory of algebraic functions, $\psi_1(x, y_g)$ is either an irreducible polynomial or a power of an irreducible polynomial. Therefore, $\psi_1(x, y)$ which when equated to zero connects the values of y, x at $a, g(a)$ respectively, is either an irreducible polynomial or a power of an irreducible polynomial. The condition that $\psi_1(x, y)$ is irreducible is contained in

THEOREM 4. *If $P(x, z_1), Q(x, z_2)$ be any two points on the Riemann surface having the same x-coordinate, the condition for $\psi_1(x, y)$ (which when equated to zero connects the values of x, y at $a, g^{-1}(a)$, g^{-1} being the inverse of the deck-transformation g) to be irreducible, is that y does not take the same value at $g^{-1}(P)$ and $g^{-1}(Q)$.*

PROOF: y takes the same value at $g^{-1}(a)$ as y_g at a. Let x take the value x at the points $P(x, z_1), Q(x, z_2), \ldots, W(x, z_m)$.

* Appell and Goursat, *Fonctions Algebriques*, p. 257.

If y_g takes the same value at more than one of these points, say at r of them, then the relation $\psi_1(x, y_g) = 0$ is reducible* and $\psi_1(x, y_g)$ is a perfect r-th power of an irreducible polynomial so that $\psi_1(x, y)$ will be a perfect r-th power of an irreducible polynomial. So for irreducibility of ψ_1, y_g must not take the same value at any two points of the sort $P, Q, \ldots, W$ at which x takes the same value, that is, for irreducibility of ψ_1, y must not take the same value at any two of the points $g^{-1}(P), g^{-1}(Q) \ldots, g^{-1}(W)$.

In the case in which the factor ψ_1, corresponding to the deck-transformation $a \rightarrow g(a)$ is *irreducible*, we bring out the essentially identical nature of the branch points of y and z considered as functions of x, by means of

THEOREM 5. *The algebraic functions of x defined by $\psi_1(x,y) = 0$ and $F(x, z) = 0$ have identical branch points of the same orders each.*

PROOF: Let S be the Riemann surface of $F(x, z) = 0$ with x as the independent variable, and let S' be the Riemann surface of $\psi_1(x, y) = 0$ also with x as the independent variable. Since $\psi_1(x, y)$, by hypothesis, is irreducible, there is a (1, 1) birational correspondence between the points of S and S'. Let a be a branch point of z *qua* function of x of order $r-1$. Then z can be expanded in the form

$$z = (x-a)^{s/r}[\alpha_0+\alpha_1(x-a)^{1/r}+\ldots].$$

Since $y=R\ (x, z)$, a rational function of z and x, y is also capable of being expanded in powers of $(x-a)^{1/r}$. It might be argued that in the substitution for z in $R(x, z)$, all powers of $(x-a)^{1/r}$ may get suppressed except those of $(x-a)^{p/r}$ where $r = p\delta$ (p, δ are integers) so that y is expressible in powers of $(x-a)^{1/\delta}$ itself. This is untenable, because then z which is a rational function of x and y will be capable of expansion in powers of $(x-a)^{1/\delta}$, or of $(x-a)^{1/\varepsilon}$ where $\delta = \gamma\varepsilon$ (γ and ε are integers) for the same reason as was considered hypothetically in the previous sentence; which in either case contradicts the hypothesis. So y is also expressible in powers of $(x-a)^{1/r}$ only, and a is a branch point of y *qua* function of x of order $r-1$.

If ψ_1 and ψ_2 be two irreducible factors of the z-resultant, the functions y of x defined by $\psi_1 = 0$, $\psi_2 = 0$ have also only identical

* Appell and Goursat, *loc. cit.*, p. 258.

branch points of the same orders since each of these functions has branch points identical with those of z and of the same orders.

Results similar to the above are also obtained for the functions z and x considered as algebraic functions of y given by $F'(y, z) = 0$; $\psi_1(x, y) = 0$.

6. *The distinctness of the factors* ψ.

Let the relation $\psi_1 = 0$ connect the values of y, x at a, $g(a)$ respectively and let $\psi_1 \equiv [\psi']^k$ so that ψ' is a polynomial of degree l/k in x and m/k in y, and k is a divisor of l and m. If $k = 1$, this becomes the case where ψ_1 is irreducible. If the value of the rational function y be denoted by $y(a)$ at a and that of the rational function x at $g(a)$ by $x(g)$, corresponding to the value $y(a)$ of y in $\psi_1 = 0$ the values of x will be $x(g) = x_1, x_2, \ldots, x_{l/k}$.

We first prove the following result.*

No two of the roots $x(g) = x_1, x_2, \ldots, x_{l/k}$ *are the values assumed by the rational function* x *at points* a, $g(a)$, *where* $a \rightarrow g(a)$ *defines a deck-transformation.*

For, the rational function y takes the value $y(a)$ at, say, $a_1, a_2, \ldots, a_l$. No two of these points, say, a, a_i define a deck-transformation $a \rightarrow a_i (= g(a))$ since in that case y and z will take the same values at two distinct points of a Riemann surface which is birationally equivalent to that of $F'(y, z) = 0$. Since $\psi_1 = 0$ connects the values of the rational functions y, x at a, $g(a)$ respectively, $x\{ g(a_2) \}$, the value of the rational function x at $g(a_2)$, should be among $x_2, \ldots, x_{l/k}$. Let $x\{ g(a_2) \} = x_2$. In the same way we assume that $x\{ g(a_3) \} = x_3, \ldots, x\{ g(a_{l/k}) \} = x_{l/k}$. $x\{ g(a_{l/k+1}) \}$ is again one of the values $x_1, x_2, \ldots, x_{l/k}$ and so on. So since no two of the a's define a deck-transformation, $x_1, x_2, \ldots, x_{l/k}$ do not contain values assumed by x at points of the sort a_k, $g(a_k)$, where g is a deck-transformation; which is the result to be proved.

We are now in a position to prove

THEOREM 6. *If* $\psi_1(x,y)$, $\psi_2(x,y)$ *be factors of the* z*-resultant corresponding to the deck-transformations* $a \rightarrow g_1(a)$, $a \rightarrow g_2(a)$ *respectively,* ψ_1 *and* ψ_2 *cannot have a non-constant common factor.*

* In this result and in the following theorem, the proof is practically the same for the case where ψ_1 is irreducible, if we put $k=1$.

PROOF: Suppose ψ_1 is a power of an irreducible polynomial so that $\psi_1 = [\psi']^k$. If ψ_2 is to have a non-constant common factor with ψ_1, the factor must be ψ' or some power of ψ' so that $\psi_2 = [\alpha\psi']^k$ where α is a constant, as otherwise ψ_2 will have distinct polynomial factors. Let the value of y at a be $y(a)$. Corresponding to the value $y(a)$ of y in $\psi_1 = 0$, the values of x are, say, $x_1, x_2, \ldots, x_{l/k}$. Since $\psi_2 = \alpha^k\psi_1$, the values of x corresponding to the value $y(a)$ of y in $\psi_2 = 0$ are also $x_1, x_2, \ldots, x_{l/k}$. From the definition of ψ_1, one of these values is $x\{g_1(a)\}$ and from that of ψ_2, one of these is again $x\{g_2(a)\}$. It follows that either at least two of the roots of the equation $\psi_1\{x, y(a)\} = 0$ are the values of the rational function x at $g_1(a)$, $g_2(a)$ i.e. at $g_1(a)$, $g_2g_1^{-1}\{g_1(a)\}$, or $x\{g_1(a)\} = x\{g_2g_1^{-1}(g_1(a))\}$. The former contradicts the result proved in the previous paragraph and the latter is untenable since then there will be two distinct points (x, z) on the Riemann surface.

7. *The Extreme Cases.*

In the case in which $F(x, z) = 0$ is Galoisian over $\Delta(z)$, the Riemann surface of $F(x, z) = 0$ with z as the independent variable is regular and admits the maximum group of n deck-transformations. The z-resultant of F, F' therefore, factorizes into the maximum number of n factors of the type mentioned in Theorem 3. *This case we call the case of maximum reducibility of* $\phi(x, y)$.

On the other hand, if $F(x, z) = 0$ has only one deck-transformation, namely, the identical one which it always has, there is only one factor of the type ψ. *Since there is always the identical deck-transformation,* $\phi(x, y)$ *has always a factor of the type* ψ.

IV

8. *Applications—Poncelet's Theorem* and its Extension.*

Here we shall examine two particular examples illustrating our main theorem of reducibility of $\phi(x, y)$.

(*a*) One is the geometrical problem of Poncelet's Theorem*.

Let x, y, z be three variable points on a conic E, such that the straight lines xz, yz touch two conics F_1, F_2 respectively.

* R. Vaidyanathaswamy, *loc. cit.*

The envelope of xy is in general a curve ϕ of class eight. Poncelet's theorem asserts that if F_1 and F_2 have the same four points p_1, p_2, p_3, p_4 in common with E, the envelope ϕ splits into two conics ϕ_1, ϕ_2 each counted twice and passing through the same four points p_1, p_2, p_3, p_4.

If now we also let x, y, z represent three points on E parametrically, then (x, z) and (y, z) are connected by two irreducible symmetric doubly quadratic relations $F_1(x, z)=0$ and $F_2(y, z)=0$. Further F_1 and F_2 are so related that they have the same discriminant points p_1, p_2, p_3, p_4. The algebraic statement of Poncelet's theorem, therefore, reduces to that the z-resultant of two irreducible symmetric double quadratics $F_1(x, z)$, $F_2(y, z)$ with identical discriminant-points splits into two other symmetric double quadratics $\phi_1(x, y)$, $\phi_2(x, y)$ with the same discriminant points as F_1, F_2.

This statement can be proved by applying our reducibility theorem as follows:

Let $$F_1 \equiv Ax^2-2Bx+C = 0$$

where A, B, C are quadratics in z, so that

$$x = (B \pm \sqrt{\Delta})/A,$$

Δ being the x-discriminant of F_1; or $x = P_0 \pm P_1\sqrt{\Delta}$, P_0, P_1 being rational functions in z. So $\sqrt{\Delta}$ is a rational function of x and z.

Similarly $$y = P'_0 \pm P'_1\sqrt{\Delta'},$$

Δ' being the y-discriminant of F_2, and P'_0, P'_1 rational functions of z obtained from F_2 in the same way as P_0, P_1 are obtained from F_1. Therefore, y is a rational function of z and $\sqrt{\Delta'}$. If $\Delta \equiv k\Delta'$ where k is a constant, which is the case when the discriminant points of F_1 and F_2 are identical, then y is a rational function of z and x. In the same way x is a rational function of z and y so that a birational correspondence is obtained between F_1 and F_2, and x, y, z are rational functions on the Riemann surface of $F_1 = 0$ with z as the independent variable. x and y are then primitive elements of the corpus of rational functions on the surface.

The roots of $F_1 = 0$ for a value z of z are $P_0 \pm P_1\sqrt{\Delta}$ and both of these are found in the corpus $K(z, \sqrt{\Delta})$ which is the algebraic extension of the corpus $K(z)$ of rational

functions of z by the adjunction of $\sqrt{\Delta}$. So this is Galoisian over $K(z)$ and hence has the maximum number of two automorphisms. It follows therefore that the corpus of rational functions on the Riemann surface, being isomorphic with $K(z, \sqrt{\Delta})$, also has two automorphisms which we are to identify with a group of two deck-transformations of the surface. The z-resultant of F_1 and F_2 therefore contains two symmetric double quadratics as factors.

(*b*) The other example is the extension of the previous one to the hyperelliptic case.

Let $F_1(x, z)$, $F_2(y, z)$ be irreducible polynomials of the second degree in x and y respectively. Let z be of degree m in F_1 and p in F_2.

The procedure of the first example holds here also.

If the discriminant points are identical, the x and y discriminants of F_1 and F_2 respectively are identical but for a constant factor. y is therefore a rational function of x and z, also x is a rational function of y and z, i.e. there is a birational correspondence between F_1 and F_2. Therefore x, y, z are rational functions on the Riemann surface of $F_1 = 0$, of which we can make z the independent variable. x and y are as before primitive elements of the corpus of rational functions on the surface.

If again Δ be the x-discriminant of F_1, $K(z, \sqrt{\Delta})$, the algebraic extension of $K(z)$ the corpus of rational functions of z by the adjunction of $\sqrt{\Delta}$ contains both roots of $F_1 = 0$ and therefore has two automorphisms. The corpus of rational functions on the surface, therefore, has a group of two automorphisms so that the surface has a group of two deck-transformations. So the z-resultant has two factors ψ_1, ψ_2, each of degree p in x and m in y.

INFINITESIMAL DEFORMATION OF RULED SURFACES

By RAM BEHARI, *University of Delhi.*

[Received 27 January 1938]

The problem of infinitesimal deformation of a ruled surface or the equivalent problem of determining a surface which corresponds to a given ruled surface with orthogonality of corresponding elements has been investigated by Goursat,* Darboux† and Haag.‡

The object of this paper is first to deal with the general problem of infinitesimal deformation of any surface and then to apply the results to the case of a ruled surface.

Let x, y, z be the coordinates of a surface S, then the coordinates of a surface S' which is an infinitesimal deformation of S are given by

$$x' = x+\varepsilon\xi, \quad y' = y+\varepsilon\eta, \quad z' = z+\varepsilon\zeta,$$

where ε is a small quantity whose second and higher powers are negligible, and ξ, η, ζ are functions of the current superficial variables u and v.

The necessary and sufficient condition for the applicability of the two surfaces S and S', viz.

$$dx'^2+dy'^2+dz'^2 = dx^2+dy^2+dz^2,$$

gives

$$dx\, d\xi+dy\, d\eta+dz\, d\zeta = 0.$$

Since ξ, η, ζ are functions of u and v, they can be regarded as the coordinates of a point on an unknown surface and then $d\xi$, $d\eta$, $d\zeta$ give an arc-element on the surface. Therefore, this equation shows that the problem of infinitesimal deformation of surfaces is the same as that of finding a surface which corresponds to a given surface with orthogonality of linear elements.

* 'Sur les equations lineaires et la methode de Laplace'. *American Journal of Mathematics*, Vol. 18; also, 'Sur les lignes asymptotiques', *Comptes Rendus* (1896), 593.

† *Lecons sur la Theorie Generale des Surfaces*, Vol. IV, p. 25.

‡ 'Sur la deformation infiniment petite des surfaces reglees' *Nouvelles Annales de Mathematique*, (1912), 312-331, 337-358.

Referred to asymptotic lines $u=$ const. and $v=$ const. as the parametric curves, the equations of the surface S are given by

$$\left.\begin{aligned} x &= \int\left(\theta_2\frac{\partial\theta_3}{\partial u}-\theta_3\frac{\partial\theta_2}{\partial u}\right)du-\left(\theta_2\frac{\partial\theta_3}{\partial v}-\theta_3\frac{\partial\theta_2}{\partial v}\right)dv, \\ y &= \int\left(\theta_3\frac{\partial\theta_1}{\partial u}-\theta_1\frac{\partial\theta_3}{\partial u}\right)du-\left(\theta_3\frac{\partial\theta_1}{\partial v}-\theta_1\frac{\partial\theta_3}{\partial v}\right)dv, \\ z &= \int\left(\theta_1\frac{\partial\theta_2}{\partial u}-\theta_2\frac{\partial\theta_1}{\partial u}\right)du-\left(\theta_1\frac{\partial\theta_2}{\partial v}-\theta_2\frac{\partial\theta_1}{\partial v}\right)dv, \end{aligned}\right\} \qquad \text{(I)}$$

where θ_1, θ_2, θ_3 are three particular solutions of

$$\frac{\partial^2\theta}{\partial u\partial v}=\lambda\theta, \qquad \text{(II)}$$

satisfying also the relation

$$\theta_1^2+\theta_2^2+\theta_3^2=\sqrt{-RR'}$$

where R and R' denote the principal radii of curvature of the surface S at (u, v), λ being any function of u and v.*

Now the surface S' which is an infinitesimal deformation of the surface S can be determined by finding ξ, η, ζ which are given by the relation

$$dx.d\xi+dy.d\eta+dz.d\zeta=0. \qquad (1)$$

This equation is equivalent to

$$\Sigma x_1\xi_1=0,\ \Sigma x_2\xi_2=0,\ \Sigma x_1\xi_2+\Sigma x_2\xi_1=0. \qquad (2)$$

Introducing a function ϕ defined by the relation

$$2V\phi=\Sigma x_1\xi_2-\Sigma x_2\xi_1,$$

we have

$$\Sigma x_1\xi_1=0,\quad \Sigma x_1\xi_2=V\phi,\quad \Sigma x_2\xi_1=-V\phi,\quad \Sigma x_2\xi_2=0 \qquad (3)$$

We know that ϕ is determined by the partial differential equation of the second order

$$\frac{\partial}{\partial u}\left(-\frac{M}{KV}\phi_2\right)+\frac{\partial}{\partial v}\left(-\frac{M}{KV}\phi_1\right)-\frac{2FM}{V}\phi=0, \qquad (4)$$

since $L=N=0$, and that

$$\xi_1=\frac{M}{KV}(X\phi_1-X_1\phi) \qquad (5)$$

and

$$\xi_2=-\frac{M}{KV}(X\phi_2-X_2\phi) \qquad (6)$$

with similar expressions for η_1, ζ_1 and η_2, ζ_2.† ξ, η, ζ are then given by

$$\int\xi_1du+\xi_2dv,\ \int\eta_1du+\eta_2dv \text{ and } \int\zeta_1du+\zeta_2dv.$$

* See Darboux, *Theorie des Surfaces*, Vol. IV, p. 24.
† Forsyth, *Diff. Geom.* pp. 402-403.

The differential equation (4) which gives ϕ can be written in a more convenient form from which three particular integrals are immediately obtained, as follows:—

Let $K = -\frac{M^2}{V^2} = -\frac{1}{\rho^2}$ where ρ is the radius of total curvature, then (4) becomes

$$\frac{\partial}{\partial u}\left(\frac{V\phi_2}{M}\right)+\frac{\partial}{\partial v}\left(\frac{V\phi_1}{M}\right)-\frac{2FM}{V}\phi = 0,$$

or $$\frac{2V}{M}\phi_{12}+\phi_2\left(\frac{MV_1-VM_1}{M^2}\right)+\phi_1\left(\frac{MV_2-VM_2}{M^2}\right)-\frac{2FM}{V}\phi = 0,$$

or $$\phi_{12}+\tfrac{1}{2}\phi_2\left(\frac{V_1}{V}-\frac{M_1}{M}\right)+\tfrac{1}{2}\phi_1\left(\frac{V_2}{V}-\frac{M_2}{M}\right)-\frac{FM^2}{V^2}\phi = 0,$$

or $$\phi_{12}+\Gamma'\phi_1+\Delta'\phi_2+FK\phi = 0. \qquad (7)$$

Here Γ', Δ', F, K are with reference to the surface S. If the corresponding quantities for the spherical representation of S be called γ', δ', f, k, then we know that $\Gamma'=-\gamma'$, $\Delta'=-\delta'$, $FK = f$, and hence we get

$$\phi_{12}-\gamma'\phi_1-\delta'\phi_2+f\phi = 0. \qquad (8)$$

But $X_{12} = \gamma'X_1+\delta'X_2-fX$, and two similar expressions for Y_{12} and Z_{12}.*

$\therefore$ X, Y, Z are three particular solutions of (8).

Since

$$-\gamma' \equiv \tfrac{1}{2}\frac{\partial}{\partial v}(\log\rho) = \frac{\partial}{\partial v}\log\sqrt{\rho} = \frac{1}{\sqrt{\rho}}\frac{\partial}{\partial v}(\sqrt{\rho})$$

and $$-\delta' \equiv \tfrac{1}{2}\frac{\partial}{\partial u}(\log\rho) = \frac{1}{\sqrt{\rho}}\frac{\partial}{\partial u}(\sqrt{\rho}),\ddagger$$

(8) can be written as

$$\phi_{12}+\frac{\phi_1}{\sqrt{\rho}}\frac{\partial}{\partial v}(\sqrt{\rho})+\frac{\phi_2}{\sqrt{\rho}}\frac{\partial}{\partial u}(\sqrt{\rho})+f\phi = 0. \qquad (9)$$

Putting $\phi\sqrt{\rho} = \theta$, we get, differentiating with regard to u,

$$\phi_1\sqrt{\rho}+\phi\frac{\partial}{\partial u}(\sqrt{\rho}) = \theta_1.$$

Differentiating with regard to v, we get

$$\phi_{12}\sqrt{\rho}+\phi_1\frac{\partial}{\partial v}(\sqrt{\rho})+\phi_2\frac{\partial}{\partial u}(\sqrt{\rho})+\phi\frac{\partial^2\sqrt{\rho}}{\partial u\partial v} = \theta_{12}.$$

* Forsyth, *Diff. Geom.* p. 261.

‡ Eisenhart, *Diff. Geom.* p. 192.

$$\therefore \quad \phi_{12}+\frac{\phi_1}{\sqrt{\rho}}\frac{\partial}{\partial v}(\sqrt{\rho})+\frac{1}{\sqrt{\rho}}\phi_2\frac{\partial}{\partial u}(\sqrt{\rho})=\frac{1}{\sqrt{\rho}}\theta_{12}-\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial u\partial v}\phi.$$

$$\therefore \quad (9) \text{ becomes } \quad \theta_{12}=\left(\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial u\partial v}-f\right)\theta \qquad (10)$$

whose three particular solutions θ_1, θ_2, θ_3, say, are given by $\theta_1=X\sqrt{\rho}$, $\theta_2=Y\sqrt{\rho}$, $\theta_3=Z\sqrt{\rho}$.

$\therefore$ ξ_1 and ξ_2 given by (5) and (6) can be expressed in the forms $\left(\theta_1\frac{\partial\theta}{\partial u}-\theta\frac{\partial\theta_1}{\partial u}\right)$ and $\left(\theta_1\frac{\partial\theta}{\partial v}-\theta\frac{\partial\theta_1}{\partial v}\right)$ respectively.

Similarly η_1, η_2, ζ_1, ζ_2 can be expressed in the forms $\left(\theta_2\frac{\partial\theta}{\partial u}-\theta\frac{\partial\theta_2}{\partial u}\right)$, $\left(\theta_2\frac{\partial\theta}{\partial v}-\theta\frac{\partial\theta_2}{\partial v}\right)$, $\left(\theta_3\frac{\partial\theta}{\partial u}-\theta\frac{\partial\theta_3}{\partial u}\right)$, and $\left(\theta_3\frac{\partial\theta}{\partial v}-\theta\frac{\partial\theta_3}{\partial v}\right)$ where θ_1, θ_2, θ_3 are three particular solutions of an equation of the form $\frac{\partial^2\theta}{\partial u\partial v}=\lambda\theta$, and θ is any other solution.

Hence the surface which corresponds to S with orthogonality of linear elements is given by

$$\left.\begin{aligned}
\xi&=\int\left(\theta_1\frac{\partial\theta}{\partial u}-\theta\frac{\partial\theta_1}{\partial u}\right)du-\left(\theta_1\frac{\partial\theta}{\partial v}-\theta\frac{\partial\theta_1}{\partial v}\right)dv,\\
\eta&=\int\left(\theta_2\frac{\partial\theta}{\partial u}-\theta\frac{\partial\theta_2}{\partial u}\right)du-\left(\theta_2\frac{\partial\theta}{\partial v}-\theta\frac{\partial\theta_2}{\partial v}\right)dv,\\
\zeta&=\int\left(\theta_3\frac{\partial\theta}{\partial u}-\theta\frac{\partial\theta_3}{\partial u}\right)du-\left(\theta_3\frac{\partial\theta}{\partial v}-\theta\frac{\partial\theta_3}{\partial v}\right)dv.^*
\end{aligned}\right\} \qquad \text{(III)}$$

Now if we choose $\quad \lambda=\frac{2}{(u+v)^2}$, and

$$\theta_1=-\frac{2a}{u+v}+a', \ \theta_2=-\frac{2b}{u+v}+b', \ \theta_3=-\frac{2c}{u+v}+c',$$

where a, b, c are functions of u only, then the surfaces given by (I) are ruled surfaces.†

When these values are substituted, (I) becomes

$$\left.\begin{aligned}
x&=\frac{2(b'c-bc')}{u+v}+\int(b'c''-b''c')du\\
y&=\frac{2(ac'-ca')}{u+v}+\int(c'a''-a'c'')du\\
z&=\frac{2(ba'-ab')}{u+v}+\int(a'b''-b'a'')du
\end{aligned}\right\} \qquad \text{(IV)}$$

* Cf. Darboux, *loc. cit.* Vol. IV, p. 25.

† See Haag, *loc. cit.* p. 317.

which are the equations of the ruled surface S referred to asymptotic lines as parametric curves.*

Also we know that the general solution of the equation

$$\frac{\partial^2\theta}{\partial u\partial v} = \frac{2}{(u+v)^2}\theta$$

is given by

$$\theta = \frac{2(A+B)}{u+v} - (A'+B')$$

where A is an arbitrary function of u only and B is an arbitrary function of v only.‡

When the values of θ, θ_1, θ_2, θ_3 are substituted in (III) we get

$$\left.\begin{aligned} \xi &= a'B' + \frac{2(aA'-a'A-aB'-a'B)}{u+v} + \int (A'a''-A''a')\,du, \\ \eta &= b'B' + \frac{2(bA'-b'A-bB'-b'B)}{u+v} + \int (A'b''-A''b')\,du, \\ \zeta &= c'B' + \frac{2(cA'-c'A-cB'-c'B)}{u+v} + \int (A'c''-A''c')\,du \end{aligned}\right\} \quad \text{(V)}$$

which are the equations of the surface S' which corresponds to the ruled surface given by (IV) with orthogonality of corresponding elements.

* The equations of a ruled surface referred to asymptotic lines have also been obtained in terms of quadratures by Lelieuvre, *Bull. Sci. Math.* (1888), 126; Villat, *Nouv. Ann. Math.* (1910), 97; and in a form free from quadratures, by Koenigs, *Comptes Rendus* (1888), p. 51.

‡ Darboux, *loc. cit.* Vol. II, p. 143.

THE ALGEBRA OF THE *e*th POWER RESIDUES

By T. VENKATARAYUDU, *Andhra University, Waltair.*

[Received 3 February 1938]

1. *Introduction.* The main object of the present paper is to show that the algebra of the Gaussian periods is identical with the algebra of the eth power residues mod p where p is a prime and e is a divisor of $p-1$. This algebra is obviously a particular case of the algebra of the eth power residues mod N where N is any integer. Since the system of residue classes mod N is abstractly identical with a cyclic group of order N, the above algebra may be interpreted to be the algebra of the eth power residues connected with a cyclic group of order N. It is therefore natural to enquire whether there is a similar algebra connected with any finite Abelian group. We shall however show by means of an example that this is not the case.

This paper may be regarded as a continuation of a recent paper* by Dr. R. Vaidyanathaswamy *viz.* "The algebra of the quadratic residues", and the notation is the same in both.

2. *The classification of elements in a finite group G.* It is well known that the conjugate classes of elements in a finite group G combine among themselves by the group operation, in the sense that if C_iC_j denotes the totality of the elements obtained by combining every element of the class C_i with every element of the class C_j by the group operation and if an element of the class C_k occurs exactly $\gamma(i, j, k)$ times in C_iC_j, then every element of C_k occurs the same number of times $\gamma(i, j, k)$ in C_iC_j. The conjugate classes of elements thus determine an algebra which is called by Dr. R. Vaidyanathaswamy the Frobenius algebra. Vaidyanathaswamy showed† that this classification of elements in a finite group is only a particular case of a more general classification of elements in the group such that the classes combine among themselves by the group operation.

THEOREM 1. *If H is any group of automorphisms of G and if an element S of G and all elements which correspond to S by H*

* *Jour. Ind. Math. Soc.* 2 (1937), 239.

† *Proc, Ind. Acad. Sc.* 5 (1937), 63.

are put into a class, the classes so formed combine among themselves by the group operation.

PROOF: If S, an element of a class B_k, occurs exactly t times in B_iB_j the element $a(S)$ must occur precisely t times in

$$a(B_iB_j) = a(B_i).a(B_j) = B_i.B_j$$

where a is any automorphism in H. By a proper choice of a, $a(S)$ can represent any element of the class B_k. (Since H is a group, the classes thus formed are exclusive). Hence the truth of the theorem.

When H is the group of inner automorphisms of G, the classes of elements of G defined as in the above theorem are the conjugate classes of elements. When H is the total group of automorphisms some of the conjugate classes are grouped together so as to form new classes which again combine among themselves by the group operation.

Vaidyanathaswamy applied the general principle of Theorem 1 to the following two particular cases, and solved the more difficult problem of actually constructing the multiplication table of the classes.

(a) The classification of the residue classes mod N defined by the total group of automorphisms of the additive cyclic group of the residue classes mod N.

(b) The classification of the residue classes mod N defined by the sub-group of the group of automorphisms formed by the quadratic residue classes mod N.

3. *The algebra of the eth power residues mod N.*

LEMMA. *The totality of the distinct elements obtained by raising every element of a finite Abelian group G to any fixed dth power forms a sub-group G(d) of G.*

PROOF: For any two elements S_1, S_2 of G

$$S_1^d.S_2^d = (S_1S_2)^d.$$

It therefore follows that if $G(d)$ is the set of all the distinct elements in G obtained by raising every element to the dth power, the product of any two elements of the set is an element in the set, for every element in the set the inverse element is also in the set, and lastly the identity element is in the set. Hence $G(d)$ is a sub-group of G.

Note. If $g(d, M)$ is the g. c. d. of d and M where M is the order of the group,

$$G\Big(g(d, M)\Big) = G(d),$$

for, if $d = d_1d_2$, where d_1 is a divisor of M and d_2 is prime to M

$$G(d_2) = G.$$

Let G be a cyclic group of order N, which we can take without loss of generality as the additive group of the residue classes mod N. The automorphisms of G correspond to the multiplication of the residue classes by a number prime to N, and the total group T of automorphisms of G is the multiplicative group of the residue classes with respect to the numbers less than N and prime to it. The order of T is therefore $\phi(N)$. A class of elements defined by T consists of all the residue classes mod N with respect to the numbers less than N and have the same *g. c. d.* with N.

If now $T(2)$ denotes the group formed by the squares of the elements of T (same notation as in the Lemma), it is clear that $T(2)$ consists of the square residue classes or the quadratic residues mod N. The algebra of the classes of elements in G defined by $T(2)$ was called by Vaidyanathaswamy *the algebra of the quadratic residues.* Per contra, if $T(e)$ is the sub-group of T formed by the eth powers of the elements of T, the algebra of the classes of elements of G defined by $T(e)$ is *the algebra of the* e^{th} *power residues.* In view of the note to Lemma 1, we take, in what follows, e as a divisor of $\phi(N)$.

4. *The algebra of the eth power residues mod p.* In this paragraph let G denote the additive cyclic group of the residue classes mod p. If g is a primitive root mod p, T is the cyclic group (multiplicative) of order $p-1$ generated by the residue class g mod p, i.e. the elements of T are

$$g \bmod p,\ g^2 \bmod p,\ g^3 \bmod p, \ldots, g^{p-1} = 1 \bmod p.$$

Let e be a divisor of $p-1$ and let $ef = p-1$.

$T(e)$ is therefore the multiplicative cyclic group generated by g^e mod p, and the elements of $T(e)$ are

$$g^e \bmod p,\ g^{2e} \bmod p, \ldots, g^{(f-1)e} \bmod p,\ g^{fe} = 1 \bmod p.$$

The classes of elements of G defined by $T(e)$ may now be taken as $B_0, C_0, C_1, \ldots, C_{e-1}$ where B_0 is the zero residue class and $C_r (r = 0, 1, \ldots, e-1)$ is the class consisting of the residue classes

$$g^r \bmod p,\ g^{r+e} \bmod p, \ldots, g^{r+(f-1)e} \bmod p.$$

This is obvious by taking any element g^r mod p in G. The elements in G which correspond to g^r mod p by the automorphisms of $T(e)$ are got by multiplying this element successively with the elements of $T(e)$. We thus obtain the class of elements in C_r.

If $C_i \oplus C_j$ denotes the totality of the elements obtained by adding each element of C_i to each element of C_j, by our general principle of Theorem 1, each element of C_k occurs the same number of times $\gamma(i, j, k)$ in $C_i \oplus C_j$, i.e.

$$\left.\begin{aligned} C_i \oplus C_j &= \Sigma\gamma(i, j, k)\ C_k + n(i, j)B_0, \\ &\qquad i, j, k = 0, 1, \ldots, e-1 \\ C_i \oplus B_0 &= C_i, \quad i = 0, 1, \ldots, e-1 \\ \text{and} \qquad B_0 \oplus B_0 &= B_0 \end{aligned}\right\} \qquad (1)$$

5. *The algebra of the Gaussian periods.* Let ρ be a primitive pth root of unity, and as before let g be a primitive root mod p and $ef = p-1$. The functions

$$\eta_0 = \rho + \rho^{g^e} + \rho^{g^{2e}} + \ldots + \rho^{g^{(f-1)e}}$$

$$\eta_1 = \rho^g + \rho^{g^{e+1}} + \rho^{g^{2e+1}} + \ldots \rho^{g^{(f-1)e+1}}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\eta_r = \rho^{g^r} + \rho^{g^{e+r}} + \ldots + \rho^{g^{(f-1)e+r}}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\eta_e = \rho^{g^{e-1}} + \rho^{g^{2e-1}} + \ldots + \rho^{g^{fe-1}}$$

are known as the f-nomial or the Gaussian periods. It is well known* that these periods combine among themselves by multiplication.

The following proof is taken from Dickson's paper†

We have

$$\eta_0\eta_k = \sum_{s=0}^{f-1} \sum_{t=0}^{f-1} \rho^{g^{es}} \cdot \rho^{g^{et+k}}$$

For a fixed s, we may replace t by $t+s$ which ranges with t over a complete set of residues mod f.

* Mathews, *Theory of Numbers*; Van der Waerden, *Moderne Algebra*, Vol. 1, p. 162

† L. E. Dickson, (1) Cyclotomy, Higher Congruences and Waring's Problem, *Amer. Jour. Math.* 57 (1935), 391.

$$\therefore \quad \eta_0\eta_k = \sum_{s=0}^{f-1}\sum_{t=0}^{f-1} \rho^{g^{es}} . \rho^{g^{e(s+t)+k}}$$

$$= \sum_{s=0}^{f-1}\sum_{t=0}^{f-1} \rho^{g^{es}(1+g^{te+k})} \qquad (A)$$

First let $et+k \equiv O(p)$. Since $0 \leqslant et+k \leqslant ef-1 \leqslant p-2$,

$$et+k = \tfrac{1}{2}(p-1).$$

If f is even, k is divisible by e, hence $k=0$, $t=f/2$. But if f is odd, k is divisible by $e/2$, while $k \neq 0$ since $ef/2$ is not divisible by e, hence $k=e/2$, $t=(f-1)/2$. Make the definition

$n_k = 1$ if f is even and $k=0$, or if f is odd and $k=e/2$

$n_k = 0$ in all other cases.

Hence $et+k \equiv O(p)$ holds for exactly n_k values of t and the corresponding part of (A) is fn_k.

Secondly, let $et+k$ be prime to p, hence $et+k$ is congruent to a power of g.

$$1+e^{gt+k} \equiv g^{ez+h} \bmod p \qquad (B)$$

where $0 \leqslant h \leqslant e-1$, $0 \leqslant z \leqslant f-1$, and h (as well as k) is fixed; let (h, k) be the number of sets of values of t and z each chosen from $0, 1, \ldots, f-1$ for which (B) holds.

Hence (h, k) is unaltered if we increase (or decrease) either k or h by any muliple of e. For fixed values of t and z satisfying (B) the corresponding part of A is

$$\sum_{s=0}^{f-1} \rho^{g^{e(s+z)+h}} = \sum_{s=0}^{f-1} \rho^{g^{es+h}} = \eta_h.$$

This completes the proof of

$$\eta_0\eta_k = \sum_{h=0}^{e-1} (k, h)\, \eta_h + f . n_k, \quad k = 0, 1, \ldots, e-1$$

Replace ρ by ρ^{g^m}. Then

$$\eta_m\eta_{m+k} = \sum_{h=0}^{e-1} (k, h)\, \eta_{m+h} + fn_k.$$

6. *The identity of the algebra of the eth power residues mod p and the algebra of the Gaussian periods.* Comparing the

periods η_r and the classes C_r, we observe that the indices of ρ in the expression for η_r are precisely the elements in the class C_r. If the η's combine among themselves by multiplication, the classes C_r combine among themselves by addition and *vice versa*. Therefore if we take $\xi_0 = 1$, we have immediately

$$\left.\begin{array}{l} \eta_i\eta_j = \Sigma\gamma(i, j, k)\eta_k + n(i, j)\xi_0 \\ \qquad \text{for } i, j, k = 0, 1, \ldots, e-1 \\ \xi_0\eta_i = \eta_i \text{ for } i = 0, 1, \ldots, e-1 \\ \text{and} \qquad \xi_0^2 = \xi_0. \end{array}\right\} \quad (2)$$

where $\gamma(i, j, k)$, $n(i, j)$ are the same in (2) and (1). We have therefore established the identity of the two algebras.

It may be noted that the problem of the determination of the constants of multiplication $\gamma(i, j, k)$ in the algebra of the Gaussian periods, still awaits complete solution. In a series of articles Dickson* determined the algebra i.e. the multiplication table $\gamma(i, j, k)$ for the particular cases $e = 8, 9, 12, 18$ and also for the case when e is a prime or double a prime. A less general problem was discussed (this too in particular cases only) by Cayley, Burnside and O. Upadhaya†, namely the determination of the cyclotomic polynomial whose roots are $\eta_0, \eta_1, \eta_2, \ldots, \eta_{e-1}$.

7. *$\gamma(i, j, k)$ is a multiplicative arithmetic function.* Let $N = p_1^{a_1}p_2^{a_2}\ldots p_r^{a_r}$ where the p's are primes. If n_1 mod $p_1^{a_1}$, n_2 mod $p_2^{a_2}, \ldots, n_r$ mod $p_r^{a_r}$ are a given set of residue classes, the residue class n mod N is uniquely determined by the Chinese Remainder theorem and is such that

$$n = n_k \bmod p_k^{a_k} \quad (k = 1, 2, \ldots, r).$$

If n is prime to N, it is clear that n_k is prime to p_k and conversely. Hence if T is the total group of automorphisms of the additive cyclic group G of the residue classes mod N and T_k is the total group of the automorphisms of the additive cyclic group G_k of the residue classes mod $P_k^{a_k}$, T is the direct compound of T_k's in the sense that any element of T is determined uniquely from a set of r elements each one from one of the

* L. E. Dickson. (1) Cyclotomy, Higher Congruences and Waring's Problem, *Amer. Jour. Math.* 57 (1935).

(2) Cyclotomy and trinomial congruences, *Trans. Amer. Math. Soc.* 37 (1935).

(3) Cyclotomy when e is composite, *Trans. Amer. Math. Soc.* 38 (1935)

† Burnside, *Proc. Lond. Math. Soc.*, 14 (1914).

component groups of automorphisms T_k and conversely, and the correspondence is established by the Chinese Remainder Theorem.

Similarly if n is an eth power of a residue class mod N, n_k is also an eth power of a residue class mod $p_k^{a_k}$ for $k=1, 2, \ldots, r$, for if

$$n \equiv t^e \bmod N$$

and

$$t \equiv t_k \bmod p_k^{a_k}$$

then

$$n_k \equiv t_k^e \bmod p_k^{a_k}.$$

It is now easy to see that $T(e)$ is the direct compound of the groups $T_k(e)$, and a class of elements in G defined by $T(e)$ is the direct compound of the classes of elements in G_k, defined by $T_k(e)$. If therefore

$$C_i = C_{1i}C_{2i}\ldots C_{ri}$$
$$C_j = C_{1j}C_{2j}\ldots C_{rj}$$
$$C_k = C_{1k}C_{2k}\ldots C_{rj}$$

are the expressions for C_i, C_j, C_k as direct compounds of the component classes

$$C_i \oplus C_j = (C_{1i} \oplus C_{1j})(C_{2i} \oplus C_{2j})\ldots(C_{ri} \oplus C_{rj})$$

and therefore $\gamma(i, j, k) = \gamma_1(i, j, k) \cdot \gamma_2(i, j, k) \ldots \gamma_r(i, j, k)$ where $\gamma_t(i, j, k)$ is the number of times the class C_{tk} occurs in $C_{ti} \oplus C_{tj}$.

Thus the function $\gamma(i, j, k)$ is a multiplicative arithmetic function of the single argument N.

8. *Relation between the constants* $\gamma(i, j, k)$ *and the number of solutions of a trinomial congruence mod* p^θ.

Let G denote the additive cyclic group of the residue classes mod p^θ. Let g be a primitive root mod p^θ. The total group T of automorphisms is the multiplicative cyclic group generated by the residue class g mod p^θ and is of order $\phi(p^\theta)$. If e is a divisor of $\phi(p^\theta)$, $T(e)$ is the multiplicative group generated by g^e mod p^θ and is of order $\phi(p^\theta)/e$.

The classes of elements in G defined by T will be $K_0, K_1, \ldots, K_{\theta-1}$ where K_t consists of all residue classes with respect to the numbers less than p^θ and have their g. c. d. with p^θ as p^t. Under the group $T(e)$ of automorphisms these classes K_t will be further divided into equinumerous sub-classes consisting of $\dfrac{\phi(p^{\theta-t})}{g(e, \phi(p^{\theta-t}))}$ elements. [The number of elements in K_t is equal

to the number of elements in G of order $p^{\theta-t}$ and is therefore equal to $\phi(p^{\theta-t})$].

A typical class of elements in G defined by $T(e)$ consists of the *distinct* elements in

$$g^r p^t \bmod p^\theta,\ g^{r+e} p^t \bmod p^\theta,\ g^{r+2e}\, p^t \bmod p^\theta, \ldots,$$
$$g^{r+\left(\frac{\phi(p^\theta)}{e}-1\right)e}.p^t \bmod p^\theta$$

which we denote by C_{rt}. In the above set each element of C_{rt} is obviously repeated

$$\frac{\phi(p^\theta)g\left(e, \phi(p^{\theta-t})\right)}{\phi(p^{\theta-t})} = p^t g\left(e, \phi(p^{\theta-t})\right) \text{ times.}$$

Hence $\gamma_{12}(i, j, k)$, the number of times the class $C_{k_1k_2}$ occurs in $C_{i_1i_2} \oplus C_{j_1j_2}$, is equal to

$$\frac{N}{p^{i_2} g\left(e, \phi(p^{\theta-i_2})\right) \times p^{j_2} g\left(e, \phi(p^{\theta-j_2})\right)}$$

where N is the number of solutions of the congruence

$$ax^e + by^e = c \bmod p^\theta,$$

a, b, c being any fixed elements of the classes $C_{i_1i_2}$, $C_{j_2j_2}$, $C_{k_1k_2}$.

9. *The case of the general Abelian group.* It is well known that the necessary and sufficient condition that the group of automorphisms of a finite Abelian group G is Abelian is that G should be cyclic. The group of isomorphisms of a non-cyclic Abelian group is non-Abelian. For a non-Abelian group our lemma in paragraph 3 is not true. However it may be possible that the lemma is true for this particular type of non-Abelian groups, namely, the groups of automorphisms of finite Abelian groups. We shall however show by means of an example that the distinct elements obtained by raising every element of a group of automorphisms of an Abelian group do not form a group.

The group of automorphisms of a finite Abelian group of type $(1, 1, \ldots, 1)$, say r in number, has been known to be the group of all matrices of rank r whose elements are the residue class mod p and whose determinants are $\not\equiv O(p)$ where p^r is the order of the group. Let us now examine the case of the Abelian group G

$$P_1^3 = 1,\ P_2^3 = 1,\ P_1P_2 = P_2P_1.$$

The group T of automorphisms of G is the group of matrices

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

where, a, b, c, d are $\equiv 0, 1, 2,$ mod 3 and $ad-bc \not\equiv 0(3)$. This group is of order 48.

It is easy to verify that the squares of the elements of T do not form a group. For example

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

are square elements of T. But

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

is not a square element of T. Hence the squares of elements of T do not form a group.

Finally I wish to thank Dr. R. Vaidyanathaswamy for suggesting this problem to me.

STATISTICS OF PHOTONS AND PLANCK'S RADIATION FORMULA

By B. N. BISWAS, *Rajaram College, Kolhapur.*

[Received 25 February 1938]

In the present paper, the statistics of Photons has been developed to derive Planck's Radiation Formula, on the line already evolved by Biswas and Sen in a previous paper*, *i.e.* by introducing the idea of discontinuity in phase space.

It was proved in that paper that the density of distribution, or more precisely, the density of an element of volume ΔT selected as the standard satisfies the differential equation

$$\sum\left(\frac{\partial\rho}{\partial q_1}\frac{\partial U}{\partial p_1}-\frac{\partial\rho}{\partial p_1}\frac{\partial U}{\partial q_1}\right)=0, \tag{1}$$

(q_1, $q_2, \ldots$, p_1, p_2, $p_3, \ldots$ denote the position and momentum coordinates respectively for the phase space) when statistical equilibrium exists among the systems of the ensemble.

The solution is

$$\rho=\frac{N}{A}e^{-\beta U}\ \text{(for Photons)} \tag{2}$$

where N is the number of systems in the ensemble, U the energy of a single system, A the arbitrary constant having the dimensions of inverse phase volume and β is a constant quantity of the same dimensions as energy. The assumption of β having the same dimensions as energy is needed in order to make the distribution independent of the units employed.

1. *Distribution of phase points in cells, whose volumes are very small compared with* ΔT *selected as the standard,* i.e. ΔT *contains a large number of cells.*

Let P denote the probability that a cell will contain one representative point, $1-P$ is the probability that the same cell will not enclose the same point. If a cell contains r R-points† it will not contain the remaining $(n-r)$ R-points where n is the number

* B. N. Biswas and D. K. Sen, *Jour. Ind. Math. Soc.*, 19 (1931-32), 130.

† R-points stand for representative points.

of R-points within the volume ΔT. Therefore, the probability that a cell will include r R-points is $P^r(1-P)^{n-r}$. The number of cells enclosing r R-points is $\alpha \times P^r(1-P)^{n-r}$, where α is the number of cells included within ΔT. $\alpha . P^{r+1}(1-P)^{n-r-1}$ is the number of cells containing $r+1$ R-points.

Therefore the ratio of the latter to the former $= P/1-P = \mu$, say. The number of cells containing $r+1$ R-points $= \mu$ times the number of cells containing r R-points. This is true for $r=0$, $r=1, \ldots$. Hence the numbers of cells containing 0 R-point, 1 R-point 2 R-points,... are respectively equal to

$$\alpha_0, \alpha_0\mu, \alpha_0\mu^2, \ldots \tag{3}$$

where α_0 is the number of empty cells.

By following the method given in the paper referred to above, it is found that

$$\mu = e^{-\beta U}. \tag{4}$$

2. *Statistics of Photons.*

Black radiation contained in a chamber at temperature T is supposed to be consisting of a large number of photons having frequencies ranging from zero to infinity, in thermal equilibrium.

A photon of frequency ν is of energy $h\nu$ having a momentum equal to $\frac{h\nu}{c}$ where c is the velocity of light.

Thus the problem of finding out the distribution of energy over various wavelengths or frequencies is equivalent to determining the distribution of photons over various frequencies.

A photon is thus represented in a phase space of six dimensions. The coordinates which fix up the position of its representative point in the phase space are the three position coordinates x, y, z and the three momentum coordinates p_x, p_y, p_z. The volume of a cell referred to before in such a phase space, is h^3.

Let us consider an element of volume ΔT situated at a point in the phase space included between the energy range $h\nu_s$ and $h(\nu_s+d\nu_s)$.

Let $\alpha_{r'} =$ total number of cells in ΔT

$$= \frac{\Delta T}{h^3}.$$

Let $p_0, p_1, p_2, \ldots, p_r$ denote the number of cells containing 0, 1, 2, 3,..., r representative points respectively.

$$\alpha_r' = \sum_{m=0}^{m=r} p_m. \tag{5}$$

From (3) and (4) we have,

$$p_r = \alpha_0 e^{-r\beta U} = \alpha_0 e^{-r\beta h\nu_s}, \; (r = 0, 1, \ldots). \tag{6}$$

Equations (5) and (6) give

$$p_r = \alpha_r' \, (1 - e^{-\beta h\nu_s}) e^{-r\beta h\nu_s}, \; (r = 0, 1, \ldots). \tag{7}$$

Let $C_0, C_1, C_2, \ldots$ represent the total number of cells in the phase space included between $h\nu_s$ and $h(\nu_s + d\nu_s)$ enclosing 0, 1, 2, 3,..., r representative points. We have

$$C_r = A_s (1 - e^{-\beta h\nu_s}) e^{-r\beta h\nu_s}, \; (r = 0, 1, \ldots) \tag{8}$$

where A_s denotes the total number of cells in the phase space within the energy layers $h\nu_s$ and $h(\nu_s + d\nu_s)$.

The total number of photons represented in the same phase space

$$= C_1 + 2C_2 + 3C_3 + \ldots$$

$$= A_s (1 - e^{-\beta h\nu_s}) \left[e^{-\beta h\nu_s} + 2e^{-2\beta h\nu_s} + 3e^{-3\beta h\nu_s} + \ldots \right]$$

$$= \frac{A_s}{e^{\beta h\nu_s} - 1}. \tag{9}$$

To calculate A_s we are required to find out the phase volume described by the photons comprised within the energy layers $h\nu_s$ and $h(\nu_s + d\nu_s)$.

This is given by $g_s = \int\int\int\int\int\int dx.dy.dz.dp_x.dp_y.dp_z$

$$= V \int\int\int dp_x.dp_y.dp_z$$

$= V \times$ volume included between two spheres of radii $\frac{h\nu_s}{c}$ and $\frac{h(\nu_s + d\nu_s)}{c}$

$$= V \times \frac{4\pi \nu_s^2 d\nu_s}{c^3} h^3, \text{ as } p_x^2 + p_y^2 + p_z^2 = \frac{h^2 \nu_s^2}{c^2}.$$

Hence the total number of cells in the particular domain of phase space is

$$\frac{V \times 4\pi \times \nu_s^2 . d\nu_s}{c^3}.$$

The number has to be doubled, as corresponding to a definite frequency and direction, there are two photons characterised by their state of polarisation from each other.

Hence
$$A_s = \frac{8\pi V \nu_s^2 \, d\nu_s}{c^3}. \tag{10}$$

From (9) and (10), we have

$$E_s = \frac{8\pi V \nu_s^2 \, d\nu_s}{c^3} \cdot \frac{h\nu_s}{e^{\beta h\nu_s} - 1}. \tag{10A}$$

Energy associated with radiation of frequency lying between ν_s and $\nu_s + d\nu_s$ per unit volume

$$= \rho\nu_s \, d\nu_s = \frac{8\pi\nu_s^2 \, d\nu_s}{c^3} \times \frac{h\nu_s}{e^{\beta h\nu_s} - 1} \tag{11}$$

which gives the density of radiation of frequency between ν_s and $\nu_s + d\nu_s$.

3. *Entropy and free energy of photons.*

In order to evaluate β and to derive expressions for the entropy and free energy of photons of frequency ν_s, it is necessary to take recourse to Boltzmann's hypothesis connecting entropy and probability.

It gives

$$S = K \log W \text{ (neglecting the constant term)} \tag{12}$$

where S denotes the entropy of the system and W the probability of the equilibrium state.

In the present case, W is assumed to give the probability of the distribution of photons of frequency ν_s among the cells in the particular domain of the phase space in the equilibrium state and is equal to the number of ways in which the same distribution is realised. Hence it is equal to

$$W = \frac{A_s!}{\pi C_r!} \tag{13}$$

where C_r gives the number of cells containing r photons.

Equations (12) and (13) give

$$S = K \log W = K \log \frac{A_s!}{\pi C_r!} = -K A_s \sum W_r \log W_r,$$

by making use of Stirling's theorem, where $W_r = \frac{C_r}{A_s}$.

$$\therefore \quad S = -KA_s \sum (1-e^{-\beta h\nu_s}) e^{-r\beta h\nu_s} \times \log (1-e^{-\beta h\nu_s}) e^{-r\beta h\nu_s},$$

from (8)

$$= -KA_s(1-e^{-\beta h\nu_s}) \times \log (1-e^{-\beta h\nu_s}) \sum e^{-r\beta h\nu_s}$$

$$+KA_s(1-e^{-\beta h\nu_s})\beta h\nu_s \sum r e^{-r\beta h\nu_s}$$

$$= -KA_s \log (1-e^{-\beta h\nu_s}) + K\beta E_s, \text{ from (10A).} \qquad (14)$$

Differentiating S with respect to E_s we get,

$$\left(\frac{\partial S}{\partial E_s}\right)_{T=\text{const.}} = K\beta = 1/T$$

from the Second Law of Thermodynamics.

Hence, $$\beta = \frac{1}{KT}. \qquad (15)$$

From (11) and (15), we get

$$\rho_{\nu_s} d\nu_s = \frac{8\pi\nu_s^2 \, d\nu_s}{c^3} \frac{h\nu_s}{e^{h\nu_s/KT}-1} \qquad (16)$$

i.e. $$\rho_\nu \, d\nu = \frac{8\pi\nu^2 \, d\nu}{c^3} \times \frac{h\nu}{e^{h\nu/KT}-1}$$

which is the same as Planck's well-known Radiation formula.

Equations (14) and (15) lead to the following,

$$S_{\nu_s} = \frac{8\pi K\nu_s^2 \, d\nu_s}{c^3} \left\{\frac{h\nu_s/KT}{e^{h\nu_s/KT}-1} - \log (1-e^{-h\nu_s/KT})\right\} \qquad (17)$$

i.e.* $$S_\nu = \frac{8\pi K\nu^2 \, d\nu}{c^3} \left\{\frac{h\nu/KT}{e^{h\nu/KT}-1} - \log (1-e^{-h\nu/KT})\right\}$$

excluding the constant term.

Free energy for the ν_s radiation

$$= F_{\nu_s} = E_{\nu_s} - TS_{\nu_s}$$

$$= \frac{8\pi\nu_s^2 KT}{c^3} \, d\nu \log (1-e^{-h\nu_s/KT}) \qquad (18)$$

from (16) and (17).

i.e. $$F_\nu{}^* = \frac{8\pi\nu^2 KT d\nu}{c^3} \log (1-e^{-h\nu/KT}).$$

* M. N. Saha and B. N. Srivastava, *A Treatise on Heat* (second edition), p. 673.

ON THE AVERAGE RADIAL INCREASE OF A CERTAIN CLASS OF INTEGRAL FUNCTIONS OF ORDER ONE AND FINITE TYPE

By V. GANAPATHY IYER.

[Received 11 February 1938]

1. Let $z = re^{i\theta} = x+iy$. We write $R(z) = x$, $I(z) = y$. Let $f(z)$ be an integral function and

$$M(r, f) = \max_{|z| \leqslant r} |f(z)|.$$

We shall say that $f(z)$ is of order one and finite type when

$$\overline{\lim_{r\to\infty}} \frac{\log M(r, f)}{r} < \infty. \tag{1}$$

If $a > 0$, we write $\log^{+} a = \max.\ (0, \log a)$ and $\log^{-} a = \min.\ (0, \log a) = -\log^{+} 1/a$. In this paper we are concerned with integral functions $f(z)$ satisfying the condition

$$\int_{-\infty}^{\infty} \frac{\log^{+}|f(x)|}{1+x^2}\, dx < \infty. \tag{2}$$

We shall say that a function $f(z)$ satisfying (1) and (2) belongs to *the class* Λ.

1.1. We denote by $n_f(r)$ the number of zeros of $f(z)$ in $|z| \leqslant r$ and by $n_f(r, \alpha, \beta)$ those lying in the region

$$|z| \leqslant r,\ \alpha \leqslant \theta \leqslant \beta, \tag{3}$$

the zeros being counted according to their multiplicity. With this notation, N. Levinson* has proved the following result.

Theorem A. *Let $f(z)$ be a function of the class Λ. Then*

(i) $\lim_{r\to\infty} \frac{n_f(r, \alpha, \beta)}{r} = 0$ *provided the angle* $\alpha \leqslant \theta \leqslant \beta$ *does not contain* $\theta = 0$ *or* π;

* *Proc. Cambridge Phil. Soc.* 31 (1935), 335-346. See also M. L. Cartwright, *Ibid.* 347-350.

(ii) *there exists a number B such that* $\lim_{r\to\infty} \frac{n_f(r)}{r} = B$;

(iii) $\lim_{r\to\infty} \frac{n_f(r, -\alpha, \alpha)}{r} = \frac{1}{2} B = \lim_{r\to\infty} \frac{n_f(r, \pi-\alpha, \pi+\alpha)}{r}$ *for each* α *in* $0 < \alpha < \pi$.

2. The object of this paper is to prove the following two results for functions of the class Λ.

THEOREM 1. *Let* $f_1(z)$ *and* $f_2(z)$ *belong to* Λ *and let* c *be a constant. Then* cf_1, f_1+f_2 *and* $f_1 f_2$ *all belong to* Λ. *If* z_0 *be fixed,* $f(z+z_0)$ *belongs to* Λ *uniformly for* $|z_0| \leqslant T$, *that is, there exists a finite* $N = N(f, T)$ *depending only on* $f(z)$ *and* T *such that**

$$\int_{-\infty}^{\infty} \frac{\log^+|f(x+z_0)|}{1+x^2}\, dx \leqslant N, \tag{4}$$

for each z_0 *in* $|z_0| \leqslant T$.

THEOREM 2. *Let* $f(z)$ *belong to* Λ. *Then*

$$\lim_{r\to\infty} \frac{1}{\log r} \int_1^r \left\{ \log |f(te^{i\alpha})| + \log |f(-te^{i\alpha})| \right\} \frac{dt}{t^2} = \pi B |\sin \alpha|, \tag{5}$$

for each α *in* $0 \leqslant \alpha \leqslant 2\pi$, *and uniformly in any angle not containing* $\theta = 0$ *or* π. *Here* B *is the number specified in* (ii) *of Theorem A*.

2.1. Theorem 1 states that Λ is a linear class and that it is closed with respect to multiplication and translation. Theorem 2 states that the average of the expression

$$\frac{\log |f(te^{i\alpha})| + \log |f(-te^{i\alpha})|}{t}, \tag{6}$$

over the interval $(1, r)$ with the weight function $1/t$ tends to a definite limit depending on α and the density of the zeros of $f(z)$. When $f(z)$ has only real zeros, (5) is an almost trivial consequence of (ii) of Theorem A, since in this case

$$\lim_{r\to\infty} \frac{\log |f(re^{i\alpha})| + \log |f(-re^{i\alpha})|}{r} = \pi B\, |\sin \alpha|, \tag{7}$$

* It will be seen from Lemma 3 that (4) implies

$$\int_{-\infty}^{\infty} \frac{|\log |f(x+z_0)||}{1+x^2} \leqslant N_1 = N_1(f, T),\ |z_0| \leqslant T. \tag{4'}$$

when $\alpha \neq 0$ or π; and (7) implies (5), while for $\alpha = 0$ or π, (5) follows from Lemma 3 below. But in the general case the limit in (7) need not exist when there are an infinity of complex zeros; but all the same, the average in (5) has a definite limit.

3. We require several preliminary lemmas. The first is the well-known Jensen's formula. The second is known as Carleman's formula and can be obtained in the form (8) by letting $\rho \to 0$ in the equation (10) in the proof of Lemma 5. We suppose throughout that $f(0) \neq 0$. If $f(z)$ has a zero of order p at $z = 0$, we have only to apply the results obtained to $z^{-p}f(z)$.

3.1. LEMMA 1. *We have, for any integral function* $f(z)$,

$$\int_0^r \frac{n_f(t)}{t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} \log |f(re^{i\theta})|\, d\theta + \log |f(0)|.$$

3.2. LEMMA 2. *Let* $z_\nu = r_\nu e^{i\theta_\nu}$, $\nu = 1, 2, \ldots$, *be the zeros of* $f(z)$ *in the half-plane* $R(z) \geqslant 0$. *Then, as* $r \to \infty$

$$\sum_{r_\nu \leqslant r} \frac{\cos\theta_\nu}{r_\nu}\left(1-\frac{r_\nu^2}{r^2}\right) = \frac{1}{2\pi}\int_1^r \{\log|f(it)| + \log|f(-it)|\}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)dt$$

$$+\frac{2}{\pi r}\int_{-\pi/2}^{\pi/2} \log |f(re^{i\theta})| \cos\theta\, d\theta + O(1).$$

3.3. LEMMA 3. *Let* $f(z)$ *be a function of* A. *Then*

$$\int_{-\infty}^{\infty} \frac{|\log |f(x)||}{1+x^2} < \infty.$$

PROOF: Applying Lemma 2 to the half-plane $I(z) \geqslant 0$, we get

$$\sum_{r_\nu \leqslant r} \frac{\sin\theta_\nu}{r_\nu}\left(1-\frac{r_\nu^2}{r^2}\right)$$

$$= \frac{1}{2\pi}\int_1^r \{\log |f(x)| + \log |f(-x)|\}\left(\frac{1}{x^2}-\frac{1}{r^2}\right)dx$$

$$+\frac{2}{\pi r}\int_0^{\pi} \log |f(re^{i\theta})| \sin\theta\, d\theta + O(1). \tag{8}$$

Noting that $\sin\theta_\nu \geqslant 0$ for $0 \leqslant \theta_\nu \leqslant \pi$, we deduce, from (8), that

$$-\tfrac{3}{4}\int_1^{r/2} \{\log^-|f(x)| + \log^-|f(-x)|\}\frac{dx}{x^2}$$

$$\leqslant -\int_1^r \{\log^-|f(x)| + \log^-|f(-x)|\}\left(\frac{1}{x^2}-\frac{1}{r^2}\right)dx$$

$$\leqslant \int_1^r \{ \log^+ |f(x)| + \log^+ |f(-x)| \} \left(\frac{1}{x^2} - \frac{1}{r^2} \right) dx$$

$$+ \frac{4}{r} \int_0^\pi \log^+ |f(re^{i\theta})| \sin\theta \, d\theta + O(1). \qquad (9)$$

The Lemma now follows from (9) in conjunction with (1) and (2), since $| \log |f(x)| | = \log^+ |f(x)| - \log^- |f(x)|$.

3.4. LEMMA 4. *Let $f(z)$ be a function of order one and finite type. Then*

$$\frac{1}{2\pi r} \int_0^{2\pi} | \log |f(re^{i\theta})| | \, d\theta = O(1), \text{ as } r \to \infty.$$

PROOF: Since $n(r) \geqslant 0$, Lemma 1 gives

$$-\frac{1}{2\pi r} \int_0^{2\pi} \log^- |f(re^{i\theta})| d\theta \leqslant \frac{1}{2\pi r} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta + \log^+ |f(0)|$$

from which the Lemma follows as in § 3.3, in virtue of (1).

3.5. LEMMA 5. *Let $f(z)$ be a function of A. Then, as $\rho \to \infty$;*

$$\log^+ |f(\rho e^{i\theta})| \leqslant \frac{\rho \sin\theta}{\pi} \left\{ \int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{x^2 - 2\rho x \cos\theta + \rho^2} dx + O(1) \right\}$$

where $\rho e^{i\theta}$ lies in $I(z) > 0$.

PROOF: Let

$$g(z, \eta) = \log \left| \frac{z - \bar{\eta}}{z - \eta} \cdot \frac{z - \dfrac{r^2}{\eta}}{z - \dfrac{r^2}{\bar{\eta}}} \right|$$

be the Green's function for the semi-circle $|z| \leqslant r$, $I(z) \geqslant 0$ with pole at $z = \eta$. Then,* for $\rho e^{i\theta}$ in this semi-circle,

$$\log |f(\rho e^{i\theta})| + \sum_{r_\nu \leqslant r} g(z_\nu, \rho e^{i\theta})$$

$$= \frac{\rho \sin\theta}{\pi} \int_{-r}^{r} \log |f(x)|$$

$$\times \left\{ \frac{1}{x^2 - 2\rho x \cos\theta + \rho^2} - \frac{r^2}{x^2 \rho^2 + 2x\rho r^2 \cos\theta + r^4} \right\} dx$$

* *Cf*, R. and F. Nevanlinna, *Acta, Sci. Fennicae*, 50, No. 5, 1-46.

$$+\frac{r^2-\rho^2}{2\pi r}\int_0^{\pi}\log|f(re^{i\theta})|$$

$$\times\left\{\frac{1}{r^2+\rho^2-2\rho r\cos(\theta-\phi)}-\frac{1}{r^2+\rho^2-2\rho r\cos(\theta+\phi)}\right\}d\phi, \tag{10}$$

where $z_\nu = r_\nu e^{i\theta_\nu}$, $\nu = 1, 2, \ldots$ are the zeros of $f(z)$ in the semi-circle. Noting that $g(z, \eta) > 0$ in the interior of the semi-circle, the Lemma follows from (10) and the Lemmas 3 and 4.

4. PROOF OF THEOREM 1. All the assertions of this Theorem except the last are obvious consequences of the properties of the function $\log^+ a$. To prove (4), let $z = x+iy$, $z_0 = x_0+iy_0$. We suppose $y_0 > 0$; $y_0 < 0$ can be treated similarly, while $y_0 = 0$ is merely the condition (2). If $\rho e^{i\theta} = x+z_0$, we get from Lemma 5,

$$\log^+|f(x+z_0)| \leqslant \frac{y_0}{\pi}\left\{\int_{-\infty}^{\infty}\frac{\log^+|f(t)|}{(t-x-x_0)^2+y_0^2}\,dt+O(1)\right\}$$

as $|x|\to\infty$, so that

$$\int_{-r}^{r}\frac{\log^+|f(x+z_0)|}{1+x^2}\,dx$$

$$\leqslant\frac{y_0}{\pi}\int_{-\infty}^{\infty}\log^+|f(t)dt\int_{-r}^{r}\frac{dx}{(1+x^2)\left[(x+x_0-t)^2+y_0^2\right]}+O(y_0), \tag{11}$$

as $r\to\infty$. Now, integrating the rational function

$$\frac{1}{(z^2+1)\left[(z+x_0-t)^2+y_0^2\right]}$$

along the boundary of the semi-circle $|z|\leqslant r$, $I(z)\geqslant 0$, we find by using the Theorem of residues that

$$\frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)\left[(t+x-x_0)^2+y_0^2\right]}=\frac{y_0+1}{(t-x_0)^2+(y_0+1)^2}. \tag{12}$$

Noting that the integrands involved in (11) are non-negative, we deduce from (12), that

$$\int_{-r}^{r}\frac{\log^+|f(x+z_0)|}{1+x^2}\,dx$$

$$\leqslant(y_0+1)\int_{-\infty}^{\infty}\frac{\log^+|f(t)|}{(t-x_0)^2+(y_0+1)^2}\,dt+O(y_0), \tag{13}$$

as $r \to \infty$. Since

$$\frac{(y_0+1)(1+t^2)}{(t-x_0)^2+(y_0+1)^2}$$

is uniformly bounded for $0 \leqslant |t| < \infty$ and $|z_0| \leqslant T$, the Theorem follows from the relation (13).

5. Proof of Theorem 2. It is obviously sufficient to prove the Theorem for $0 \leqslant \alpha < \pi$. The case $\alpha = 0$ follows from Lemma 3. So we suppose $0 < \alpha < \pi$. We shall write, for shortness,

$$P = P(t, \alpha) = \log |f(te^{i\alpha})| + \log |f(-te^{i\alpha})|$$

and $P^+ = \max. (0, P)$ and $P^- = \min. (0, P)$. We apply Lemma 2 to the half-plane $R(ize^{-i\alpha}) \geqslant 0$. If $z_\nu = r_\nu e^{i\theta_\nu}$, $\nu = 1, 2, \ldots$, are the zeros of $f(z)$ in this half-plane, we get

$$S(r) = \sum_{r_\nu \leqslant r} \frac{\cos\left(\theta_\nu - \alpha + \frac{\pi}{2}\right)}{r_\nu}\left(1 - \frac{r_\nu^2}{r^2}\right)$$

$$= \frac{1}{2\pi}\int_1^r P(t, \alpha)\left(\frac{1}{t^2} - \frac{1}{r^2}\right)dt$$

$$+ \frac{2}{\pi r}\int_{\alpha-\pi}^{\alpha} \log |f(re^{i\theta})| \sin(\alpha - \theta)\, d\theta + O(1) \tag{14}$$

as $r \to \infty$. Choose δ such that

$$\left.\begin{aligned} &0 < \delta < \min.\left(\alpha, \pi - \alpha, \left|\alpha - \frac{\pi}{2}\right|\right), \text{ if } \alpha \neq \frac{\pi}{2}; \\ &0 < \delta < \frac{\pi}{2}, \text{ if } \alpha = \frac{\pi}{2}. \end{aligned}\right\} \tag{15}$$

We split up the sum $S(r)$ in (14) into two parts $S_\delta^{(1)}(r)$ and $S_\delta^{(2)}(r)$. The sum $S_\delta^{(1)}(r)$ corresponds to those terms of $S(r)$ for which $|\theta_\nu| \leqslant \delta$, and $S_\delta^{(2)}(r)$ corresponds to the remaining terms.

We have

$$S_\delta^{(1)}(r) = H(\alpha, \delta, r) \cdot \sum_{\substack{r_\nu \leqslant r \\ |\theta_\nu| \leqslant \delta}} \frac{1}{r_\nu}\left(1 - \frac{r_\nu^2}{r^2}\right)$$

$$= H(\alpha, \delta, r)\int_1^r \frac{1}{t}\left(1 - \frac{t^2}{r^2}\right)dn_f(t, -\delta, \delta) + O(1)$$

$$= H(\alpha, \delta, r)\int_1^r \left(\frac{1}{t^2}+\frac{1}{r^2}\right) n_f(t,-\delta, \delta)dt+O(1), \qquad (16)$$

where $H(\alpha, \delta, r)$ satisfies the conditions

$$K(\alpha, \delta) \leqslant H(\alpha, \delta, r) \leqslant L(\alpha, \delta), \qquad (17)$$

the functions K and L being defined by

$$\left.\begin{aligned} K(\alpha, \delta) &= \begin{cases} \sin(\alpha-\delta), & \alpha < \frac{\pi}{2} \\ \cos\delta, & \alpha = \frac{\pi}{2} \\ \sin(\alpha+\delta), & \alpha > \frac{\pi}{2} \end{cases} \\ L(\alpha, \delta) &= \begin{cases} \sin(\alpha+\delta), & \alpha < \frac{\pi}{2} \\ 1, & \alpha = \frac{\pi}{2} \\ \sin(\alpha-\delta), & \alpha > \frac{\pi}{2} \end{cases} \end{aligned}\right\} \qquad (18)$$

Similarly

$$\begin{aligned} 0 &< S_\delta^{(2)}(r) \\ &\leqslant \int_1^r \frac{1}{t}\left(1-\frac{t^2}{r^2}\right) d\left\{ n_f(t, \delta, \alpha)+n_f(t, \alpha-\pi,-\delta) \right\}+O(1) \\ &\leqslant \int_1^r \left(\frac{1}{t^2}+\frac{1}{r^2}\right)\left\{ n_f(t, \delta, \alpha)+n_f(t, \alpha-\pi,-\delta) \right\}dt+O(1) \\ &= O(\log r) \qquad (19) \end{aligned}$$

in virtue of (i) of Theorem A. Using (iii) of Theorem A in (16) and then combining (16), (17) and (19), we get

$$\tfrac{1}{2} BK(\alpha, \delta) \leqslant \overline{\lim_{r\to\infty}} \frac{S(r)}{\log r} \leqslant \tfrac{1}{2} BL(\alpha, \delta). \qquad (20)$$

Since the middle terms in (20) are independent of δ we get, by letting $\delta \to 0$ in (20),

$$\lim_{r\to\infty} \frac{S(r)}{\log r} = \tfrac{1}{2} B \sin\alpha, \qquad (21)$$

in virtue of the relations (18). Combining (14), (21) and using Lemma 4, we conclude that

$$\lim_{r\to\infty} \frac{1}{\log r}\int_1^r P(t, \alpha)\left(\frac{1}{t^2}-\frac{1}{r^2}\right)dt = \pi B \sin\alpha. \qquad (22)$$

We have now to show that (22) implies (5). For this purpose we shall first establish two further Lemmas.

5.1. LEMMA 6. *The relation* (22) *implies that*

$$\int_1^r |P(t,\alpha)|\frac{dt}{t^2} = O(\log r).$$

PROOF. Proceeding as in Lemma 3, we get, from (22), that

$$\begin{aligned} -\tfrac{3}{4}\int_1^{r/2} P^- \frac{dt}{t^2} &\leqslant -\int_1^r P^-\left(\frac{1}{t^2}-\frac{1}{r^2}\right)dt \\ &\leqslant \int_1^r P^+\left(\frac{1}{t^2}-\frac{1}{r^2}\right)dt + O(\log r) \\ &= O(\log r) \qquad (23) \end{aligned}$$

in virtue of (1). The lemma follows from (23) as in § 3.3.

5.2. LEMMA 7. *The relation* (22) *implies that*

$$\int_{r/2}^r |P(t,\alpha)|\left(\frac{1}{t^2}-\frac{1}{r^2}\right)dt = o(\log r).$$

PROOF. We have, from (22),

$$\frac{1}{\log r}\int_1^r P(t,\alpha)\left(\frac{1}{t^2}-\frac{1}{r^2}\right)dt - \frac{1}{\log\frac{r}{2}}\int_1^{r/2} P(t,\alpha)\left(\frac{1}{t^2}-\frac{4}{r^2}\right)dt = o(1). \qquad (24)$$

Now (24) can be written

$$\begin{aligned} -\frac{1}{\log r}\int_{r/2}^r P^-\left(\frac{1}{t^2}-\frac{1}{r^2}\right)dt &= o(1) + \frac{1}{\log r}\int_{r/2}^r P^+\left(\frac{1}{t^2}-\frac{1}{r^2}\right)dt \\ &\quad + \left(\frac{1}{\log r}-\frac{1}{\log\frac{r}{2}}\right)\int_1^{r/2} P\left(\frac{1}{t^2}-\frac{1}{r^2}\right)dt + \frac{3}{r^2\log\frac{r}{2}}\int_1^{r/2} P\,dt \\ &\leqslant o(1)+o(1)+\frac{3}{r^2\log\frac{r}{2}}\int_1^{r/2} P^+ dt, \qquad (25) \end{aligned}$$

in virtue of (1) and (22). Using (1) again in (25) and proceeding as in § 3.3, we arrive at the result required.

5.3. We can now complete the proof of Theorem 2. The relation (24) gives [as in (25)]

$$\begin{aligned} \frac{3}{r^2\log\frac{r}{2}}\int_1^{r/2} P\,dt &= o(1) - \frac{1}{\log r}\int_{r/2}^r P\left(\frac{1}{t^2}-\frac{1}{r^2}\right)dt \\ &\quad + \left(\frac{1}{\log\frac{r}{2}}-\frac{1}{\log r}\right)\int_1^{r/2} P\left(\frac{1}{t^2}-\frac{1}{r^2}\right)dt \\ &= o(1)+o(1)+o(1) = o(1) \qquad (26) \end{aligned}$$

in virtue of (22) and Lemma 7. Replacing r by $2r$ in (26) we get

$$\int_1^r P\,dt = o(r^2 \log r)$$

which in combination with (22) gives (5). So the theorem is proved.

Postscript. The fact of uniformity stated in the enunciation of Theorem 2 is not brought out by the proof given in § 5—5·3. The following line of proof will remove this defect. The sum $S(r)$ in (14) could be written as

$$\sin\alpha \sum_{r_\nu \leqslant r} \frac{\cos\theta_\nu}{r_\nu}\left(1-\frac{r_\nu^2}{r^2}\right) - \cos\alpha \sum_{r_\nu \leqslant r} \frac{\sin\theta_\nu}{r_\nu}\left(1-\frac{r_\nu^2}{r^2}\right). \qquad (14)'$$

From the relation (8) it is easily proved that the series

$$\sum \frac{|\sin\theta_\nu|}{r_\nu}$$

converges even when we consider all the zeros of $f(z)$.

Hence instead of (21), it is sufficient to show that

$$U(r) = \sum_{r_\nu \leqslant r} \frac{\cos\theta_\nu}{r_\nu}\left(1-\frac{r_\nu^2}{r^2}\right) \sim \tfrac{1}{2}B \log r \qquad (21)'$$

uniformly for α in $0 < \lambda \leqslant \alpha \leqslant \mu < \pi$. Let δ be chosen so that $\delta < \min.(\lambda, \pi-\mu)$. Let $U_\delta^{(1)}(r)$ denote the sum of those terms of (21)′ corresponding to $|\theta_\nu| \leqslant \delta$, and $U_\delta^{(2)}(r)$ the sum of the remaining terms. It is seen as in § 5, that $U_\delta^{(1)}(r)$ lies between

$$\int_1^r \left(\frac{1}{t^2}+\frac{1}{r^2}\right) n_f(t, -\delta, \delta)\,dt + O(1)$$

and $\cos\delta$ times the same quantity. $U_\delta^{(2)}(r)$ can be treated as before, and we arrive at (21)′ exactly as (21). The rest of the proof proceeds as before.

STUDIES IN TURBINE GEOMETRY—I
THE CONCEPTS OF TURBINE GEOMETRY

By A. NARASINGA RAO, *Annamalainagar.*

[Received 20 July 1938]

"Wie fruchtbar ist der kleinste Kreis
Wenn Man ihn wohl zu pflegen weiss!"
GOETHE: Sprüche.

This paper is devoted to the creation of the necessary concepts required for the building up of what promises to be an important extension of circle and sphere geometries with the help of a fruitful idea due to Prof. Edward Kasner—the TURBINE. I have, however, changed somewhat the significance of this term so that the two-dimensional turbine may have the same features as its confreres in higher space, and Kasner's turbine is here called a "semi-turbine", since two of them go to make up a turbine according to my terminology.

The turbine is essentially an association of an oriented (Laguerre) sphere and a non-oriented (Mobius) sphere in concentric position, so that it serves as a connecting link between the geometry of non-oriented spheres (Niedere Kugelgeometrie) and that of oriented spheres (Höhere Kugelgeometrie). Its introduction into sphere-geometry has as its first fruits not only a possibility of restating familiar results so that they acquire a richer content, but also a keener insight into the relation between Mobius- and Laguerre-Geometries and of a whole pencil of geometries (in the sense of Klein's Erlanger Program) which lie in between.

1. A sphere* in a n-dimensional Euclidean space is said to be "oriented" if a definite sign is attached to its radius when it is a real number, opposite signs corresponding to "opposite orientations". The orientation may be associated with fixing a definite sense to the normal to the sphere and hence to all its surface elements—say, towards the centre if the radius is considered positive and the opposite way if the radius be negative.

* I use the word 'sphere' in the sense of 'circle' in a plane and 'hypersphere' in n-space. The 'surface element' means similarly the tangent-line element in a plane and the tangent prime element in n-space.

A surface element or a sphere is said to be non-oriented if it is oriented both ways. (1.1)

The angle between two surface elements taken in a particular order is the rotation which will carry the first into the second in such a manner that an orientation of the first agrees with an orientation of the second. It will be seen from the above definition that *when both the surface elements are oriented the angle between them is known mod 2π, but when either or both are non-oriented the angle is determinate mod π.* (1.2)

2. *The Turbine.*

The dominant concept of the new treatment is the 'turbine' which is defined as follows:

DEF: *Let every oriented tangential surface element of a sphere c in n-space glide in all directions along the tangent prime through the same distance t. The resulting configuration of oriented surface elements will be called an oriented turbine* and denoted by the symbol (c, t).*

The points of the surface elements thus obtained lie on a concentric sphere C of radius R, while the elements make an angle α with the surface element of C where

$$R^2 = r^2+t^2\,; \quad \cos^2\alpha = r^2/R^2. \tag{2.1}$$

Since R is given through its square, we shall regard the "Mobius sphere" or "outer sphere" or "rim" C of the turbine as non-oriented, while the "Laguerre sphere" or "inner sphere" or "hub" c is an oriented sphere. However, as C is the carrier of oriented surface elements, an orientation may be induced on C by laying down that the oriented normal to C must make an acute angle [see (1.2)] with the oriented normals to the "surface elements" it carries. This is equivalent to orienting C by giving its radius the same sign as that of c. When this has been done, α in (2.1) is determinate mod. 2π, and $\cos\alpha = r/R$.

The turbine admits of an 'inner' or 'hub' specification (c, t) and an 'outer' or 'rim' specification (C, α). If we keep the rim C fixed but vary α, we get an *outer pencil* of turbines which when

* The word TURBINE is due to Kasner (*Amer. Jour. Math.* 1911 and *Trans. Amer. Math. Soc.* July 1937) and may be defined as the aggregate of line elements of an oriented circle each made to glide through a constant distance *in the direction of orientation.* This configuration which is peculiar to two dimensions I call "semi turbine". My definition of n-dimensional turbine becomes in 2-dimensions Kasner's turbine with the proviso in italics dropped out and comprises two of Kasner's "semi-turbines".

$\alpha = 0$ or π becomes an oriented sphere. Similarly if c is kept fixed but t is allowed to vary, we have an *inner pencil* of turbines which, when $t = 0$, becomes the oriented sphere c, and for $t = \infty$ will be later identified with the unique non-oriented point sphere at infinity of Höhere Kugelgeometrie.

An inner pencil of turbines is an aggregate of ∞^{2n-2} surface-elements which, apart from the orientation, are determined by a partial differential equation of the first order and second degree whose Cauchy characteristics are straight lines which touch c.

Two oriented spheres are said to have *proper contact* if they have an oriented surface element in common. Two turbines are said to "touch" if the locus of points common to their outer spheres is incident with the envelope of primes properly tangent to their inner spheres.

3. *Sub-Geometries of Turbine Geometry.*

The turbine concept belongs essentially to the geometry of surface elements and is a generalisation of the oriented sphere which is a special turbine for which $t = 0$, or $\alpha = 0 \bmod \pi$, or alternatively whose rim and hub coincide ($C = c$).

The totality of turbines in S_n is an ∞^{n+2} aggregate and may be represented by points in an S_{n+2}, the oriented spheres corresponding to points on a quadratic variety Ω^2_{n+1} in S_{n+2}. General Turbine Geometry G_T is the invariant theory of transformations of turbines into turbines induced by the projective group in the representative S_{n+2}. In general the inner and outer spheres will themselves be transformed into turbines. (3.1)

Corresponding to the sub-group of the projective group in S_{n+2} which keeps Ω invariant, we have as a sub-geometry the Lie Geometry G_L of turbine transformations which carries oriented spheres into oriented spheres and conserves proper contact not only between spheres but also between turbines. This implies that it may be considered* as a transformation of surface elements into surface elements and turbines as aggregates of surface elements into turbines. Neither the inner nor the outer sphere of a turbine will in general be carried over into the inner or outer sphere of the new turbine. (3.2)

Laguerre Geometry G_l may be imbedded in Turbine geometry as the invariant theory of the sub-group of Lie transformations which carry every inner pencil into an inner pencil. The outer

* This is mentioned by Kasner (for $n=2$) in his papers referred to above

sphere will not, in general, be carried over into the outer sphere of the transformed turbine. (3.3)

Again, Mobius Geometry G_m is the invariant theory of the restricted group of Lie transformations which carry every outer pencil into an outer pencil. The inner sphere will not in general be carried over into the inner sphere. Since the two oriented spheres which are otherwise identical but differ in orientation belong to the same outer pencil, they are carried over into another sphere oriented both ways. Hence, according to (1.1) we may say that non-oriented spheres are carried over into non-oriented spheres. (3.4)

Finally we have the Euclidean geometry G_e in S_n corresponding to turbine transformations in G_L which carry outer pencils into outer pencils, and also inner pencils into inner pencils. The radius of a circle is an invariant in this geometry. (3.5)

4. *Representation of turbines in S_n by points S_{n+2}.*

Let c be a sphere in a Euclidean S_n whose centre, referred to rectangular axes, is $\xi_1, \xi_2, \ldots, \xi_n$ and whose radius (oriented) is r. We represent the turbine (c, t) by the point in a projective S_{n+2} whose homogeneous coordinates are

$$\left.\begin{aligned} x_0 &= \tfrac{1}{2}\rho(1+\xi_1^2+\ldots\ldots+\xi_n^2-r^2-t^2) \\ x_1 &= \tfrac{1}{2}\rho(1-\xi_1^2-\ldots\ldots-\xi_n^2+r^2+t^2) \\ x_{i+1} &= \rho\xi_i \quad (i = 1, 2, \ldots, n) \\ x_{n+2} &= \rho r \end{aligned}\right\} \qquad (4.1)^*$$

so that when the turbine becomes a sphere ($t = 0$), the corresponding point (x) lies on the Absolute quadratic†

$$\Omega(x, x) \equiv -x_0^2+x_1^2+\ldots+x_{n+1}^2-x_{n+2}^2 = 0.$$

In the same manner, the turbine whose outer specification is (C, α), C being concentric with c and of radius R, is represented by the point whose coordinates are

$$\left.\begin{aligned} x_0 &= \tfrac{1}{2}\rho(1+\xi_1^2+\ldots\ldots+\xi_n^2-R^2) \\ x_1 &= \tfrac{1}{2}\rho(1-\xi_1^2-\ldots\ldots-\xi_n^2+R^2) \\ x_{i+1} &= \rho\xi_i \quad (i = 1, 2, \ldots, n) \\ x_{n+2} &= \rho R \cos\alpha \end{aligned}\right\} \qquad (4.2)$$

* *Vide* a paper 'On a principle of duality in circle, sphere, and Line Geometries', *Math. Zeitschrift* Bd. 44 (1938) p. 192, § 8 where the coordinates (4.1) and (4.2) are used, but without a clear realisation of their geometrical significance.

† BLASCHKE: *Vorlesungen uber Differentialgeometrie* Bd. III S. 227.

which when $\cos\alpha = \pm 1$ gives a point on Ω. A comparison of (2.1), (4.1) and (4.2) shows that to each turbine whether defined through its hub or its rim specification corresponds a unique point in S_{n+2}.

By varying t in the equation (4.1), we see that *to the turbines of an inner pencil correspond points on a straight line in S_{n+2} passing through the point* $(-1, 1, 0, 0, \ldots, 0)$. (4.3)

Similarly by varying α in (4.2) we have the result; *to the turbines of an outer pencil correspond points on a straight line S_{n+2} through the point* $(0, 0, \ldots, 0, 1)$. (4.4)

The turbine $(-1, 1, 0, 0, \ldots, 0)$ common to all inner pencils satisfies $\Omega = 0$ and may be considered as a sphere. A change of orientation is brought about by a change in the sign of x_{n+2}, and this leaves it unaltered; hence it is oriented both ways, that is, non-oriented. Since it is a sphere $t = 0$, and an examination of (4.1) shows that it may be taken as the limit when the radius tends to infinity of a sphere with its centre at an arbitrary point ξ_i. We call it "the point sphere at infinity" and shall denote it hereafter by the letter l, since it is an invariant point for Laguerre geometry G_l. (4.5)

The turbine $(0, 0, \ldots, 0, 1)$ common to all outer pencils is the pole of the prime $x_{n+2} = 0$—which represents the totality of all turbines of inner radius zero including in particular, all point spheres—with respect to Ω. It may be considered as the limit of a turbine (c, t) with any centre when $r \to \infty$ and t^2 is negative and such that $r^2 + t^2$ is finite. Its outer specification would correspond to an indeterminate C with $\cos\alpha \to \infty$. It must be considered non-oriented. It will be represented by the letter m since it is an invariant point for the Mobius group G_m. (4.6)

Since a change in the orientation of a turbine corresponds to a change in the sign of r and therefore of x_{n+2}, it follows that two turbines or spheres which differ only in their orientation are represented by points collinear with m and separating harmonically m and its polar prime $x_{n+2} = 0$. Hence points on $x_{n+2} = 0$ are oriented both ways, that is, they correspond to non-oriented turbines with the same rim sphere as the two oriented spheres in the outer pencil. *In imbedding Mobius Geometry in Turbine Geometry we regard these non-oriented turbines* $(C, \alpha = \pi/2)$ *as*

the analogues of the non-oriented sphere. Hence, using the same terminology we may say:

Two spheres which differ only in their orientation correspond to points on Ω collinear with m. The same sphere non-oriented may be represented by the section of the line with $x_{n+2} = 0$. The points common to Ω and $x_{n+2} = 0$ correspond to spheres of zero radius. (4.7)

It is thus clear that Mobius geometry of non-oriented spheres corresponds to the sub-group of projective transformations in S_{n+2} which transforms Ω into itself and carries $x_{n+2} = 0$ and hence also its pole m into themselves. Obviously lines through m are carried over into lines through m, which proves (3.4). Similarly, in Laguerre geometry oriented spheres are transformed into oriented spheres, those of infinite radius (primes) being transformed into primes. Since spheres of infinite radius correspond to the section of Ω by $x_0+x_1 = 0$, we have Ω as an invariant variety and $x_0+x_1 = 0$ as an invariant prime. Hence its pole l is a fixed point, and inner pencils which correspond to lines through l are carried over into inner pencils, which proves (3.3).

From (4.2), (4.3) and (4.4) we infer:

If a turbine is represented in S_{n+2} by a point P, we get the point in S_{n+2} corresponding to its hub c (oriented) by taking the intersection, other than l of lP with Ω. The rim sphere C (non-oriented) is represented by the intersection of mP with $x_{n+2} = 0$. (4.8)

5. The following results may be verified by actually carrying out the calculations:—

If x_i and $y_i (i = 0, 1, \ldots, n+2)$ are the coordinates in S_{n+2} of two oriented spheres, so that they satisfy $\Omega(x, x) = 0$, $\Omega(y, y) = 0$, and if θ is their angle of intersection and t_{12} the length of their common tangential segment, then

$$\cos\theta = (-x_0y_0+x_1y_1+\ldots\ldots+x_{n+1}y_{n+1})/x_{n+2}y_{n+2} \quad (5.1)$$

$$-t_{12}^2 = \frac{2(-x_0y_0+x_1y_1+\ldots+x_{n+1}y_{n+1}-x_{n+2}y_{n+2})}{(x_0+x_1)(y_0+y_1)}$$

$$= 2\Omega(x, y)/(x_0+x_1)(y_0+y_1). \quad (5.2)$$

Two points x, y possess with respect to the automorphic group of a quadric $\Omega(x, x) = 0$ an absolute invariant

$$\Omega(x, y)^2/\Omega(x, x)\Omega(y, y).$$

If x, y represent two turbines $(c_1, t_1) = (C_1, \alpha_1)$, and $(c_2, t_2) = (C_2, \alpha_2)$, it will be found that

$$\frac{4\Omega(x, y)^2}{\Omega(x, x)\,\Omega(y, y)} = \frac{(-2x_0y_0+2x_1y_1+\ldots+2x_{n+1}y_{n+1}-2x_{n+2}y_{n+2})^2}{(-x_0^2+\ldots+x_{n+1}^2-x_{n+2}^2)(-y_0^2+\ldots+y_{n+1}^2-y_{n+2}^2)}$$
$$= (t_1^2+t_2^2-t_{12}^2)^2/t_1^2\,t_2^2$$
$$= 4(\cos\theta-\cos\alpha_1\cos\alpha_2)^2/\sin^2\alpha_1\sin^2\alpha_2$$

whence t_{12} is the length of the common tangential segment of the two inner spheres c_1, c_2 and θ the angle of intersection of the two outer spheres C_1C_2 when oriented similarly to c_1c_2 respectively.

The conjugacy relation $\Omega(x, y) = 0$ is invariant for transformations of the Lie Group. Hence

Two turbines for which $t_1^2+t_2^2 = t_{12}^2$ or alternatively $\cos\theta = \cos\alpha_1\cos\alpha_2$ are called conjugate turbines. Conjugacy is an invariant relation in G_L and hence also in its sub-geometries. (5.3)

When one of the turbines is a sphere, the conjugacy relation implies that the turbine has a surface element in common with the oriented sphere. (5.4)

The condition of proper contact of two turbines as defined in **2** is easily shown to be $t_{12}^2 = (t_1 \pm t_2)^2$, that is

$$1 = \frac{t_1^2+t_2^2-t_{12}^2}{4t_1^2t_2^2} = \frac{\Omega(x, y)^2}{\Omega(x, x)\,.\,\Omega(y, y)}. \qquad (5.5)$$

Since this is an invariant relation for G_L, we have proved (3.2) that two turbines which touch, and in particular two oriented spheres which have proper contact, are carried over into turbines or spheres related in the same way. Now a surface element may be associated with the totality of spheres through it, and these are carried over into spheres with a common surface element. Thus G_L is a transformation of oriented surface elements in S_n which carries oriented spheres into oriented spheres.

The polarity defined by Ω associates with each S_r in S_{n+2} a conjugate S_{n+1-r} which cuts Ω in a V_{n-r}^2. Thus every S_r may be considered as the locus of points conjugate to $n-r+2$ points on Ω. Hence by (5.4)

An S_r in the Turbine Space S_{n+2} represents the totality of turbines which touch $n-r+2$ oriented spheres. (5.6)

In particular, the points on a plane (S_2) section of Ω through l represent oriented spheres in S_n touching n oriented primes and hence inscribed in a cone of revolution having their common point

for vertex. Let us call such a system of spheres a *cotangential system* since the tangent primes of any two members touch all other members. Now take any straight line in S_{n+2}. This represents a pencil of turbines whose inner spheres are, according to (4.8), represented by the section of Ω by the plane through l and the line; and whose outer spheres are represented by a line in $x_{n+2}=0$. Hence

A straight line in turbine space represents a "pencil of turbines". The inner spheres of such a pencil belong to a cotangential system while the outer spheres belong to a coaxial system. A turbine is formed by the association of concentric members, one of each system. (5.7)

The points on Ω in S_{n+2} may be made to correspond to oriented spheres in S_n, and also by (4.7) to point spheres in S_{n+1}. The representation thus obtained of oriented spheres in S_n by points in S_{n+1} will be found on examination to be the extension to higher space of Laguerre's *isotropic projection.**

When $n = 3$, there is a third interpretation for points on Ω as representing lines in S_3, the correspondence thus obtained between lines and oriented spheres in S_3 being Lie's famous "Gerade-Kugel Transformation". The turbine concept gives it a richer significance as a special case of "Complex-Turbine Transformation" by which to a linear complex corresponds an oriented turbine. Two complexes in involution correspond to two turbines which are "conjugate", and special linear complexes (straight lines) to special turbines (oriented spheres). However, as the signature of Plücker's Absolute is $+ + + - - -$, while that of Ω is $+ + + + - -$, the transformation does not carry real lines or complexes into real spheres or turbines.

6. A quadric in S_4 may be considered as a prime section of Plücker's Absolute, and hence its points correspond to lines in S_3 belonging to a linear complex Γ. We thus have three interpretations for points on Ω in S_4: (*i*) as oriented circles in a plane, (*ii*) as lines in a projective 3-space with an invariant complex Γ, and (*iii*) as point spheres in pentaspherical 3-space. The imaginary circle at infinity ω is not a concept of pentaspherical space, but where its use helps to shorten and clarify the description, it is used and the statement appears in italics. The details of the correspondence are given below:

* LAGUERRE: *Oeuvres* Tome II, 592-670.
KLEIN: *Vorlesungen uber Höhere Geometrie*, Dritte Auflage, S. 246.

Plane S_2 of Höhere Kreisgeometrie.	Projective S_3 with an invariant complex Γ.	Pentaspherical S_3; *Proj.* S_3 *with an invariant conic* ω.	Projective S_4 with an invariant quadric Ω.
1. Oriented Circle	Line of Γ.	Point sphere.	Point on Ω.
2. Oriented Turbine *s*.	Linear Complex *s* in involution with Γ.	Non-oriented sphere *s*; *Quadric through* ω.	Point *s* not on Ω.
3. Two turbines which are "conjugate" (*vide* 5.3).	Two complexes in involution with each other and with Γ.	Two orthogonal spheres	Two points conjugate with respect to Ω.
4. Oriented circles through an oriented line element.	Pencil of lines belonging to Γ; a surface element of Γ.	Pencil of point spheres; *isotropic line*.	A generator of Ω.
5. Two oriented line elements which lie on the same circle.	Two united surface elements of Γ i.e. the plane of each passes through the point of the other.	*Two intersecting isotropic lines.*	Two intersecting generators of Ω.
6. A family of turbines in contact (i.e. through an oriented line element).	Linear complexes containing a pencil of lines of Γ.	*Spheres through a given isotropic line.*	Polar plane of a generator *w.r.t.* Ω.
7. Kasner's "Turbine" (Semi-turbine according to our terminology)	Pencils of the Congruence Γ*s* which have their vertices on one of their common conjugate lines.	One system of generating lines of a sphere *s*.	One regulus of the section of Ω by the polar prime of *s*.

7. *Outer coordinates of a Turbine.*

Given $n+2$ oriented spheres $A_1, \ldots, A_{n+2}$, we may define two intrinsic systems of coordinates with the help of the concepts of of Mobius and Laguerre geometry as follows:—We obtain the outer pencil of a given turbine (C, α) by replacing $\cos\alpha$ by $X_r + \cos\alpha$ in (4.2) and find the unique member which touches A_r i.e. which is conjugate to A_r (*vide* 5.4). If θ_r is the angle of intersection of the outer sphere of the turbine (oriented similarly to its inner sphere) and A_r, then by (5.3) we have $X_r = \cos\theta_r - \cos\alpha$. We take the numbers X_r to be the *outer coordinates* of the turbine (C, α). It will be found that in the system of coordinates used in §§ **4**, **5** we have

$$X = \cos\theta - \cos\alpha = \frac{\Omega(x, a)\cos\alpha}{x_{n+2}a_{n+2}} \tag{7.1}$$

where x, a are respectively the coordinates of the turbine and one of the reference spheres A. Hence

If θ_r be the angle at which the outer sphere C of a turbine (C, α) cuts the reference spheres A_r, the $n+2$ numbers $X_r = \cos\theta_r - \cos\alpha$ are the outer coordinates of the turbine. These are non-homogeneous coordinates in S_{n+2} as may be seen by (7.1). (7.2)

When the turbine becomes an oriented sphere, $\alpha = 0$ and $X_r = \cos\theta_r - 1 = -2\sin^2\frac{1}{2}\theta_r$. If we prefer it, we may take $\xi_r = \cos\theta_r$ as the actual coordinates, since this implies only a change of origin. If this is done we call these coordinates "the cosine coordinates of the oriented sphere". These cosine coordinates satisfy a quadratic relation since the corresponding point must lie on Ω. This is given by the Frobenius identity

$$\begin{vmatrix} 1 & c_{12} & \cdot\cdot & \cdot\cdot & \xi_1 \\ c_{21} & 1 & \cdot\cdot & \cdot\cdot & \xi_2 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ c_{n+2,1} & \cdot\cdot & \cdot\cdot & 1 & \xi_{n+2} \\ \xi_1 & \xi_2 & \cdot\cdot & \xi_{n+2} & 1 \end{vmatrix} = 0, \tag{7.3}$$

or equivalently $\Sigma c^{rs}\xi_r\xi_s = 1$ where $c_{rs} = c_{sr}$ is the cosine of the angle between A_r and A_s, and $\Sigma c^{rs}c_{rt} = 0$ or 1 according as $s \neq t$ or $s = t$.

If the orientation of one of the reference spheres be reversed, the corresponding cosine coordinate changes sign, and hence it is equivalent to a linear transformation applied to the coordinates.

When the sphere is of zero radius, its cosine coordinates are infinite. Hence,

the condition that the sphere whose cosine coordinates are ξ_r be of zero radius is $\Sigma c^{rs}\xi_r\xi_s = 0$. (7.4)

From (7.1) we have, when $\cos\alpha \rightarrow 0$,

$$\cos\theta_1 : : : \cos\theta_{n+2} = \frac{\Omega(x, a^{(1)})}{a_{n+2}^{(1)}} : : : \frac{\Omega(x, a^{(n+2)})}{a_{n+2}^{(n+2)}}. \quad (7.5)$$

Now a sphere of Mobius geometry is a turbine with vanishing inner radius $\left(\therefore \cos\alpha = 0 \text{ by } (2.1)\right)$ and corresponds to a point on the prime $x_{n+2} = 0$. Since the expressions on the right of (7.5) are linear and homogeneous in $x_1, x_2, \ldots, x_{n+1}$ it follows that *the ratios of the cosine coordinates $\xi_1 : \xi_2 : : : \xi_{n+2}$ of a Mobius sphere may be considered as the projective conrdinates of its representative point in $x_{n+2} = 0$. The equation of the Absolute of the Mobius space $x_{n+2} = 0$ is $\Sigma c^{rs}\xi_r\xi_s = 0$ as seen in (7.4).* (7.6)

If the orientation of a sphere be reversed, all its cosine coordinates change their sign so that the ratios are unaffected.

It is possible to give a geometrical interpretation of the coefficients c^{rs} as follows: Let $A'_1, A'_2, \ldots, A'_{n+2}$ be spheres orthogonal to sets of $n+1$ of the spheres $A_1, \ldots, A_{n+2}$ obtained by omitting one at a time, and let these be oriented any way. Let γ_{rs} be the cosine of the angle between A'_r and A'_s, and let ω_r be the angle between A_r and A'_r. Then it may be easily verified that $c^{rr} = \sec^2\omega_r$ and $c^{rt}/\sqrt{(c^{rr}c^{tt})} = \gamma_{rt}$. Hence *the identical relation satisfied by cosine coordinates may be thrown into the form*

$$\frac{\xi_1^2}{\cos^2\omega_1} + \ldots\ldots + \frac{2\gamma_{12}\xi_1\xi_2}{\cos\omega_1\cos\omega_2} + \ldots\ldots = 1. \quad (7.7)$$

If the orientation of, say, A'_1 be reversed, γ_{1r} and $\cos\omega_1$ change their signs so that (7.7) remains unaffected.

8. *Tangential or Inner Coordinates of a Turbine.*

Again, given any turbine (c, t) we may form its inner pencil by replacing t^2 in (4.1) by $T_r + t^2$ and determine the unique member of the pencil conjugate to A_r. If t_r is the length of the tangential segment common to c and A_r, we have by (5.3), $T_r = t_r^2 - t^2$. We take the $(n+2)$ numbers T_r as the tangential or

inner coördinates of the turbine (c, t). It will be found that in the system of coordinates used in §§ **4**, **5**, we have

$$T_r = t_r^2 - t^2 = \frac{-2\Omega(x, a^{(r)})}{(x_0+x_1)\ (a_0^{(r)}+a_1^{(r)})} \tag{8.1}$$

Hence,

if t_r is the common tangential segment of the inner sphere c of a turbine (c, t) and the reference sphere A_r, the numbers $T_r = t_r^2 - t^2$ are the inner coordinates of the turbine. They are non-homogeneous coordinates in S_{n+2} as may be seen from (8.1). If $n+3$ reference spheres $A_1, \ldots, A_{n+3}$ are used, the ratios of $T_1 : : : : T_{n+3}$ form the projective coordinates of the representative point of the turbine in S_{n+2}. (8.2)

Connecting the $n+3$ actual ordinates we have the following identity

$$\begin{vmatrix} 0 & t_{12}^2 & \cdot\cdot & t_{1,n+3}^2 & T_1 \\ t_{21}^2 & 0 & \cdot\cdot & t_{2,n+3}^2 & T_2 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ t_{n+3,1}^2 & \cdot\cdot & \cdot\cdot & 0 & T_{n+3} \\ T_1 & T_2 & \cdot\cdot & T_{n+3} & -t^2 \end{vmatrix} = 0 \tag{8.3}$$

which serves to express t^2 in terms of the actual coordinates $T_1, \ldots, T_{n+3}$.

If $t = 0$ the turbine reduces to an orientated sphere, and the representative point lies on Ω. Putting $t = 0$ in (8.3), we have a homogeneous quadratic in T_r which is thus the equation of Ω in T_r-coordinates.

Postscript on the contact of two turbines.

We may say that two turbines "touch" if the join of their representative points in S_{n+2} touches Ω. The following geometrical relations then hold between the two turbines:

(i) The two spheres in the pencil determined by the two given turbines coincide. (*vide* 5.7)

(ii) The intersection of the two outer spheres is incident with the envelope of primes having proper contact with the inner spheres. [This is the definition given in § **2**]

(iii) If p is a prime having proper contact with the two inner spheres, then the sections by p of the two outer spheres have contact.

(iv) If P is a common point of the two outer spheres, then the two enveloping cones from P to the two inner spheres have proper contact (along a generator).

(v) If t_{12} is the common tangential segment of the two outer spheres, we have $t_{12}^2 = (t_1 \pm t_2)^2$ [*vide* 5.5].

(vi) If one or both of the turbines reduce to spheres, the contact between the two turbines implies contact between the spheres and the turbine, or between the two spheres, in the sense of having an oriented surface element in common.

COMPARISON OF SANNIA'S THEORY OF LINE CONGRUENCES WITH GAUSS'S THEORY OF SURFACES

By RAM BEHARI, *University of Delhi.*

[Received 27 January 1938]

1. The relation between the theory of line congruences and Gauss's theory of surfaces has been noted by Kummer.* He mentions the analogy between Hamilton's formula $p = p_1 \cos^2\theta + p_2 \sin^2\theta$ and Euler's formula $\frac{1}{R} = \frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2}$. The analogy between the total and mean curvatures of the surface, and the total and mean parameters of the congruence and some other analogies have been mentioned by Sannia.†

The object of this paper is to point out some other similarities between the two theories.

2. (i) Corresponding to the five families of ruled surfaces found by K. Ogura‡ through a line of the congruence, *viz.*

(*a*) the surfaces whose spherical representations are minimal lines, whose equation is $Edu^2 + 2Fdudv + Gdv^2 = 0$,

(*b*) the developable surfaces whose equation is

$$\delta du^2 + 2\delta' dudv + \delta'' dv^2 = 0,$$

(*c*) the surfaces of distribution, whose equation is

$$J(f, \phi) = 0,$$

(*d*) the principal surfaces given by $J(f, J(f, \phi)) = 0$,

(*e*) the characteristic surfaces given by $J(\phi, J(f, \phi)) = 0$,

where $f \equiv Edu^2 + 2Fdudv + Gdv^2$ and $\phi \equiv \delta du^2 + 2\delta' dudv + \delta'' dv^2$ are Sannia's two fundamental quadratic forms§, we have the five families of curves on a surface, *viz.*

* Allgemeine Theorie der geradlinigen strahlensysteme, *Journal fur Mathematik*, 1860, 226-230.

† Nuova esposizione della geometria infinitesimale della congruenze rettilinee, *Annali di Matematica* 1908, 159.

‡ 'Differential Geometry of the Line Congruence', *Science Reports of Tohoku Imp. Univ.* 1916, p. 114.

§ See Bianchi, *Lezioni di geometria differenziale*, Vol. I, p. 459.

(a') null lines given by $Edu^2+2Fdudv+Gdv^2=0$,

(b') the asymptotic lines given by

$$Ldu^2+2Mdudv+Ndv^2=0,$$

(c') the lines of curvature given by $J(\psi_1, \psi_2)=0$,

(d') the curves given by $J(\psi_1, J(\psi_1, \psi_2))=0$,

(e') the characteristic curves given by

$$J(\psi_2, J(\psi_1, \psi_2))=0,$$

where E, F, G; L, M, N are the fundamental magnitudes of the first and second order respectively, and

$$\psi_1 \equiv Edu^2+2Fdudv+Gdv^2;$$

$$\psi_2 \equiv Ldu^2+2Mdudv+Ndv^2.$$

(ii) If two directions on the sphere conjugate with the developable directions make angles θ, θ' with the curve $v=$ const., which represents a surface of distribution, we have

$$\tan\theta \tan\theta' = -P_1/P_2\text{*}.$$

This corresponds in Gauss's Theory to the theorem† :—

If two conjugate directions of the surface make angles θ, θ' with a principal direction, then

$$\tan\theta \tan\theta' = -\rho_2/\rho_1.$$

(iii) For any elliptic congruence, there is a unique conjugate system of ruled surfaces through a line of the congruence for which the angle between the directions on the sphere is minimum; it is the only conjugate system whose directions are symmetric with respect to the directions of the distributive surfaces.‡

For the corresponding theorem in Gauss's Theory, see Eisenhart *Diff. Geom.*, p. 130.

(iv) If θ be the angle which one characteristic direction on the sphere makes with the surface of distribution $v=$ const., the other characteristic direction makes an angle $-\theta$. Hence the parameters of distribution of the characteristic surfaces are equal.

* K. Ogura, *loc. cit.* p. 112.

† A. J. MacConnell, *Applications of the Absolute Differential Calculus*, p. 216.

‡ K. Ogura, *loc. cit.* p. 111. This is the system of characteristic ruled surfaces.

The necessary and sufficient conditions that the characteristic surfaces be represented by parametric curves are $\frac{\delta}{E}=\frac{\delta''}{G}$; $\delta'=0$.

For the corresponding theorem in Gauss's Theory, see Eisenhart *loc. cit.*, p. 130.

(v) The necessary and sufficient conditions that the developable directions be parametric are $\delta=0$, $\delta''=0$. Also, the developable surfaces are self-conjugate.

Corresponding to these in Gauss's Theory are the theorems :-

The necessary and sufficient conditions that the asymptotic lines be parametric are $L=0$, $N=0$.

The asymptotic lines are self-conjugate.

(vi) If the total parameter K is zero, *i.e.* if $\delta\delta''-\delta'^2=0$, the congruence may be called parabolic. The lines of the congruence are the tangents to a system of asymptotic lines of a surface.*

In Gauss's Theory, when $K=0$ we get $LN-M^2=0$, i.e. parabolic points.

(vii) If p is the parameter of distribution of a surface whose spherical representation makes an angle θ with the curve $v=$ const. representing a surface of distribution, we have Hamilton's formula

$$p=p_1\cos^2\theta+p_2\sin^2\theta,$$

where p_1, p_2 are the principal parameters of distribution.†

If p' is the parameter of distribution of another surface whose spherical representation is orthogonal to that of the above surface, we have

$$p'=p_1\sin^2\theta+p_2\cos^2\theta.$$

∴ by addition we get $p+p'=p_1+p_2=$ const.

Hence the sum of the parameters of distribution of two ruled surfaces whose directions on the unit sphere are orthogonal is constant.

This corresponds in Gauss's Theory to the theorem :—

'Sum of the normal curvatures in two orthogonal directions is constant'.

Cf. Zindler, *Liniengeometrie*, Vol. II, p. 108.

† Bianchi, *Geometria Differenziale*, Vol. I, p. 462, or Zindler *loc. cit.*, p. 80.

(viii) If p and p' be the parameters of distribution of two conjugate ruled surfaces through a line of the congruence, then we have

$$\frac{1}{p}+\frac{1}{p'}=\frac{1}{p_1}+\frac{1}{p_2};\ \frac{\sin\omega}{pp'}=\frac{1}{p_1p_2},$$

where ω is the angle between the two conjugate directions.*

The first relation shows that the sum of the reciprocals of the parameters of distribution of two ruled surfaces whose spherical representations are in conjugate directions is constant.

The second relation shows that the product of the parameters of distribution of the ruled surfaces whose spherical representations are in conjugate directions is a maximum for distributive surfaces and a minimum for characteristic surfaces.

These results correspond in Gauss's Theory to the following known results:—

The sum of the normal radii at a point in conjugate directions is constant. The product of the normal radii in conjugate directions is a maximum for characteristic lines and a minimum for lines of curvature.

(ix) If θ, θ' be the angles which two directions which are conjugate with the developable directions make with the direction $v=$ const. which represents a surface of distribution, we have

$$\tan\theta\tan\theta'=-p_2/p_1,$$

$$\therefore\qquad \tan\omega\equiv\tan(\theta-\theta')=\frac{p_2+p_1\tan^2\theta}{(p_1-p_2)\tan\theta}.$$

If ω is constant, say equal to $\tan^{-1}c$, we have

$$p_1\tan^2\theta-(p_1-p_2)c\tan\theta+p_2=0,$$

which shows that corresponding to a given angle ω_1 there are two ruled surfaces through a line for which $\omega=\omega_1$.

If they are orthogonal, the congruence is normal; also for all congruences for which $p_1/p_2=$ const., these surfaces are those which cut the surfaces of distribution at a constant angle.†

Corresponding to these results we get in Gauss's Theory the following theorems:—

Corresponding to a given angle ω_1, two curves can be drawn through a point on a surface, which will make angles with their

* Sannia, *loc. cit.*, p. 154.

† B. Su, 'Certain Double Systems of Ruled Surfaces', *Japanese Journal of Mathematics* 1927, p. 211.

conjugates equal to ω_1. If these curves are orthogonal, the surface is a minimal surface. Also for all surfaces for which $\rho_2/\rho_1 =$ const., these curves are those which cut the lines of curvature at a constant angle.

(x) *If the coefficients in the quadratic forms*

$$Edu^2+2Fdudv+Gdv^2 \text{ and } \delta du^2+2\delta' dudv+\delta'' dv^2$$

satisfy the three equations

$$-\gamma(EG-F^2)^{\frac{1}{2}} = \delta_2-\delta_1'-\Gamma'\delta+(\Gamma-\Delta')\delta'+\Delta\delta'', \qquad \text{(I)}$$

$$\gamma'(EG-F^2)^{\frac{1}{2}} = \delta_1''-\delta_2'+\Gamma''\delta+(\Delta''-\Gamma')\delta'-\Delta'\delta'', \qquad \text{(II)}$$

$$(EG-F^2)^{-\frac{1}{2}}\left(-\frac{\partial\gamma}{\partial v}+\frac{\partial\gamma'}{\partial u}\right)+\frac{E\delta''+G\delta-2F\delta'}{EG-F^2}=0, \qquad \text{(III)}$$

where $\gamma \equiv \sum X x_1, \gamma' \equiv \sum X x_2$; (x, y, z) are the coordinates of a point on the surface of reference and X, Y, Z the direction cosines of the line through it, then they determine a congruence the Cartesian equations of whose middle surface can be obtained by quadratures.*

This corresponds to Bonnet's Theorem in Gauss's Theory. [See Forsyth, *Diff. Geo.* p. 50].

(xi) If ρ_1 and ρ_2 are the focal distances of a line, we have $\rho_1\rho_2 = \cos\theta \dfrac{dS}{d\sigma}$, where θ is the angle which the ray meeting the surface of reference at $M(x, y, z)$ makes with the normal to the surface of reference, dS is the element of area of the surface of reference at M and $d\sigma$ is the element of area of the spherical representation.†

If the congruence is normal, we have $\theta = 0$, $\cos\theta = 1$, and hence $\rho_1\rho_2 = \dfrac{dS}{d\sigma}$ which corresponds to the well known theorem of Gauss.‡

* Bianchi, *loc. cit.*, p. 497; Sannia, *loc. cit.* p. 166; *Math. Ann.* 1910, p. 414.

† *Cf.* Strazzeri, Sulle Congruenze di rette, *Palermo Rendiconti* 1927, p. 139.

‡ Eisenhart *Diff. Geo.* p. 145.

ON THE ZEROS OF WEIERSTRASS'S NON-DIFFERENTIABLE FUNCTION*

By C. N. SRINIVASIENGAR, *Bangalore.*

1. The zeros of certain standard non-differentiable functions have been the subject-matter of a few papers—mostly of an introductory and numerical type. For Weierstrass's non-differentiable function

$$\sum_{0}^{\infty} a^n \cos b^n \pi x,$$

where $0 < a < 1$, and b is an odd integer such that $ab > 1 + \frac{3\pi}{2}(1-a)$, the values $x = m + \frac{1}{2}$ where m is any integer obviously give zeros. Ganesh Prasad† verified that these are limiting points of the zeros of the function. Thus, taking $a = \frac{1}{2}$, $b = 13$, he verified that at least one zero of the function occurs in each of the intervals

$$\begin{array}{ll} (\frac{1}{2} \pm 1/13^k), & (\frac{1}{2} \pm \frac{3}{2}/13^k) \\ (\frac{1}{2} \pm \frac{3}{2}/13^k), & (\frac{1}{2} \pm 2/13^k) \\ (\frac{1}{2} \pm 3/13^k), & (\frac{1}{2} \pm \frac{7}{2}/13^k) \\ (\frac{1}{2} \pm \frac{7}{2}/13^k), & (\frac{1}{2} \pm 4/13^k), \text{ etc} \end{array}$$

where k is any positive integer. He gave similar intervals for the general case, taking $b = 4m+1$ and $b = 4m-1$ separately.

On the other hand, A. N. Singh‡ has shown that for his function (an extended form of Peano's function), every zero is a limiting point of the zeros. That such a property is *nearly* true for Weierstrass's function will be pointed out in this paper. Ganesh Prasad evidently did not consider this possibility. Singh

* Read at the Tenth Conference of the Indian Mathematical Society at Lucknow (March 1938). Theorem 2 has been modified later into its present form.

† *Proc. Benares Math. Soc.* XI (1929), 1-8.

‡ *Bull. Calcutta Math. Soc.* XXII (1930), 91-98.

also writes,* "It would be interesting to find some special character of the set of zeros, *e.g.* whether they form an enumerable set or not, or whether the set is closed or open".

2. Let $f(x)$ denote any continuous everywhere non-differentiable function. If ξ be a number such that

$$D^{+}f(\xi)=D^{-}f(\xi)=+\infty\ ;\ D_{+}f(\xi)=D_{-}f(\xi)=-\infty,$$

then ξ is called a *knot-point.* Knot-points constitute the general type, but exceptional points will always exist. We have the theorem of W. H. Young†, which states that there necessarily exists a set of points which is everywhere dense and is of the first category such that at any point of this set there is a distinction of right and left in the values of the derivates. The exceptional set may in particular contain a sub-set of points η such that

$$D^{+}f(\eta)=D_{+}f(\eta)=\pm\infty\ ;\ D^{-}f(\eta)=D_{-}f(\eta)=\mp\infty.$$

Such points are called *cusps,* and if this subset of cusps exists, its measure is known to be zero. Non-differentiable functions are known to exist, however, which have no cusps. Peano's function and Singh's generalisation of it are examples.

We now prove

THEOREM 1. *If ξ is a zero of any non-differentiable function $f(x)$, and if ξ is a knot-point, ξ is necessarily a limiting point of the zeros of $f(x)$.*

If possible, suppose ξ is an isolated zero of $f(x)$. Then we can find a positive number k such that in the interval $(\xi-h,\ \xi+h)$ where $0 < h < k$, $f(x)$ has no zero other than ξ. Now, two cases are conceivable:

(i) $f(\xi+h)$ and $f(\xi-h)$ are of the same sign,‡ say positive. $\therefore$ $f(\xi+h)/h$ is positive and $f(\xi-h)/-h$ is negative. Hence $D_{+}f(\xi)\geqslant 0$, so that ξ is not a knot-point.

(ii) $f(\xi+h)$ and $f(\xi-h)$ are of opposite signs, say $f(\xi+h)$ is positive, and $f(\xi-h)$ is negative. Therefore $f(\xi+h)/h$ and

* A. N. Singh. *The Theory and Construction of Non-Differentiable Functions* (Lucknow University Studies, 1935). The question of enumerability of the zeros, and properties of the set of limiting points still require elucidation.

† *Messenger of Mathematics* 38 (1909), 65-69.

‡ It is easy to see that the *converse* of the property, "If $f(x)$ is continuous in (a, b), and if $f(a)$ and $f(b)$ are of unlike signs, $f(x)$ vanishes once at least in (a, b)" is not true for functions other than polynomials. In other words, a continuous function $f(x)$ may be always $\geqslant 0$ in (a, b) and possess any number of roots in (a, b).

$f(\xi-h)/-h$ are both positive. Hence the relations $D_+ f(\xi) = D_- f(\xi) = -\infty$ are impossible, so that ξ is not a knot-point.

COROLLARY. *More generally, if ζ is any knot-point of $f(x)$, the equation $f(x) = f(\zeta)$ has ζ as a limiting point of its roots. In other words, there are an infinite number of roots of the equation $f(x) = f(\zeta)$ in any interval enclosing ζ.*

3. For Weierstrass's function

$$\phi(x) = \sum_{0}^{\infty} a^n \cos b^n \pi x$$

where a and b satisfy the requisite inequalities, the points M/b^m where m is any positive integer, and M is any integer (including zero) are cusps. These points determine all the cusps. At any other point, G. C. Young* has proved that *at least on one side the upper derivate is $+\infty$, the lower derivate is $-\infty$, while on the other side one at least of the derivates is infinite.*

Suppose now that Weierstrass's function can possess a zero ξ which is not a cusp, and which is not a limiting point of the zeros. Proceeding as in 2, if case (i) presents itself, $\phi(\xi+h)/h$ cannot, as $h \to +0$, have an upper limit $+\infty$ and lower limit $-\infty$; similarly $\phi(\xi-h)/-h$ cannot, as $h \to +0$, have $+\infty$ and $-\infty$ as upper and lower limits. If case (ii) presents itself, $\phi(\xi+h)/h$ and $\phi(\xi-h)/-h$ are of the sign for all h in $0 < h < k$, so that neither $\phi(\xi+h)/h$ nor $\phi(\xi-h)/-h$ can have $+\infty$ and $-\infty$ as upper and lower limits, as $h \to +0$. The existence of an isolated zero ξ at a point other than a cusp is therefore in contradiction with the theorem of G. C. Young. Hence we have

THEOREM 2. *The zeros of Weierstrass's non-differentiable function, other than those that may happen to be cusps, are limiting points of the zeros.*

A similar result at once follows for the equation $\phi(x) = \phi(\xi)$ where ξ is any point other than a cusp of $\phi(x)$.

Ganesh Prasad's result that $x = \frac{1}{2}$ is a limiting point of the zeros follows at once, since $x = \frac{1}{2}$ is not a cusp.

It is possible that a cusp can be a zero of $\phi(x)$. Thus, taking $m = 1$, $\phi\left(\frac{M}{b}\right)$ will vanish if $\cos \frac{\pi M}{b} + (-)^M \frac{a}{1-a} = 0$. Hence, if M is odd,

* *Quarterly Journal of Mathematics* 47 (1916), p. 165.

$$a = \frac{\cos \frac{\pi M}{b}}{1 + \cos \frac{\pi M}{b}}$$

The integers b and M can be chosen so as to satisfy the inequality for ab. For example, $M = 3$, $b = 17$.

If a cusp M/b^m be a zero of $\phi(x)$, we should have

$$\left\{ \cos \frac{\pi M}{b^m} + a \cos \frac{\pi M}{b^{m-1}} + \ldots + a^{m-1} \cos \frac{\pi M}{b} \right\} + (-1)^M \frac{a^m}{1-a} = 0.$$

We might consider the cusps in the interval (0, 1) and take $0 < M < b^m$. Given M, m and b, this determines an equation of degree m in a. This equation must possess at least one real root lying between 0 and 1 and satisfying the inequality $ab > 1 + \frac{3\pi}{2}(1-a)$. If we now have these values of a and b, the occurrence of a zero of the function at another cusp $M'/b^{m'}$ requires that a should satisfy a *similar* equation of degree m'. It appears therefore unlikely that for any values of a and b, we can have in the interval (0, 1) an infinite number of zeros which are cusps of the corresponding $\phi(x)$. A formal proof of this may be desirable.

Lastly, it must be noted that *a zero which may occur at a cusp is necessarily isolated.** For, suppose a cusp η of the non-differentiable function $f(x)$ is a limiting point of the zeros, $f(\eta)$ being also zero. Let $\eta + h_1$, $\eta + h_2, \ldots$ be a sequence of zeros on one side of η, where $h_n \to 0$. Then $\frac{f(\eta+h)-f(\eta)}{h} \to 0$ as h tends to zero through the set $h_1, h_2, \ldots h_n, \ldots$ But the definition of a cusp requires that $\frac{f(\eta+h)-f(\eta)}{h}$ tends to either $+\infty$ or $-\infty$.

4. Theorem 2 will evidently be true for any non-differentiable function for which the property proved by G. C. Young for Weierstrass's function can be extended. A consideration of Peano's function will however show that this property is *not true* for all non-differentiable functions.

* This remark was made to me by Dr. A. N Singh from geometrical considerations.

CONVERGENCE THEOREMS FOR SERIES OF POSITIVE TERMS

By C. T. RAJAGOPAL, *Christian College, Madras.*

[Received 23 April 1938]

1. This paper brings together certain theorems which enable us to settle the convergence or divergence of $\sum^{\infty} a_n$, $(a_n > 0)$ with the help of a suitably chosen integral $\int^{\infty} F(x)\,dx$, $F(x) > 0$. The theorems include as particular cases all the convergence criteria, in common use, for series of positive terms, and suggest a simple and not inelegant method of presenting such criteria within the framework of a single idea.

Theorems 1 and 3 involve the nth term of the series and may be said to be of "the first kind", adopting a classification of du Bois Reymond. Theorems 2 and 4 are companion theorems of "the second kind" and involve the ratio a_{n+1}/a_n.

2. THEOREM 1. *If* (i) $\sum^{\infty} a_n$ *is a series of positive terms,* (ii) $\sum^{\infty} d_n$ *is another such series with* $d_n = O(1)$ *and* $D_n \equiv \sum_{\nu=1}^{n} d_\nu \to \infty$, (iii) $F(x)$ *is positive monotone decreasing, then* $\sum^{\infty} a_n$ *is convergent* [*or divergent*], *provided*

(iv) $\dfrac{a_n}{d_n} \leqslant F(D_n)$ [*or* $\geqslant F(D_n)$],

(v) $\int^{\infty} F(x)dx$ *is convergent* [*or divergent*].

PROOF. The result follows from the fact that $\sum^{\infty} d_n F(D_n)$ converges or diverges with $\int^{\infty} F(x)dx$.(3)

2.1. By a method which I have indicated elsewhere,(5) the proof of Theorem 1 may be generalized so that it leads to

THEOREM 2. *If (i), (ii) are as in Theorem 1, (iii) $f(x)$ has a continuous derivative $f'(x)$ which is such that $\int^{\infty}|f'(x)|dx$ is convergent, then $\sum^{\infty} a_n$ is convergent [or divergent], provided*

$$\text{(iv)}\quad \frac{1}{d_n}\log\frac{a_{n+1}d_n}{a_n d_{n+1}} \leqslant f(D_n)\ [\text{or} \geqslant f(D_n)],$$

$$\text{(v)}\quad \int^{\infty} e^{\int^{x} f(t)dt}\,dx \text{ is convergent [or divergent].}$$

The following proof of the theorem is more direct than my original proof.(5)

PROOF. If the first alternatives of (iv), (v) are given,

$$\log\frac{a_{\nu+1}d_\nu}{a_\nu d_{\nu+1}} \leqslant \int_{D_{\nu-1}}^{D_\nu} f(t)dt + \int_{D_{\nu-1}}^{D_\nu}\{f(D_\nu)-f(t)\}d(t-D_{\nu-1}).$$

Integrating by parts the second term of the right hand member, we have

$$\log\frac{a_{\nu+1}d_\nu}{a_\nu d_{\nu+1}} \leqslant \int_{D_{\nu-1}}^{D_\nu} f(t)dt + \int_{D_{\nu-1}}^{D_\nu}(t-D_{\nu-1})f'(t)dt.$$

Summing for $\nu = m+1, m+2, \ldots, n-1$ and observing that

$\sum^{\infty} j_\nu \equiv -\sum^{\infty}\int_{D_{\nu-1}}^{D_\nu}(t-D_{\nu-1})f'(t)dt$ is convergent (absolutely),

we obtain

$$\log\frac{a_n d_{m+1}}{a_{m+1}d_n} < \int_{D_m}^{D_{n-1}} f(t)dt + K_1 \qquad (K_1 \text{ fixed}).$$

Again, since $|f(t)| < M$ (fixed), $d_n < K$ (fixed),

$$-KM < \int_{D_{n-1}}^{x} f(t)dt \qquad \text{for } D_{n-1} \leqslant x \leqslant D_n.$$

From the last two inequalities, by addition

$$\log\frac{a_n d_{m+1}}{a_{m+1}d_n} < \int_{D_m}^{x} f(t)dt + K_1 + KM \qquad (D_{n-1} \leqslant x \leqslant D_n);$$

or,

$$\frac{a_n}{d_n} < \frac{a_{m+1}}{d_{m+1}} e^{K_1+KM} e^{\int_{D_m}^{x} f(t)dt}, \qquad (D_{n-1} \leqslant x \leqslant D_n)$$

and so

$$a_n < \frac{a_{m+1}}{d_{m+1}} e^{K_1+KM}\int_{D_{n-1}}^{D_n} e^{\int_{D_m}^{x} f(t)dt}\,dx.$$

A comparison of the given series $\sum^{\infty} a_n$ with the convergent series $\sum^{\infty} \int_{D_{n-1}}^{D_n} e^{\int_{D_m}^{x} f(t)dt}\, dx$ leads to the desired result.

The divergence part of the theorem is similarly proved.

2.2. Deductions from Theorems 1, 2. (*A*) Taking $F(x) = e^{-\rho x} (\rho > 0)$ and hence $f(x) = -\rho$, we see that $\sum^{\infty} a_n$ is convergent when

$$(1):C \qquad \frac{\log \dfrac{a_n}{d_n}}{D_n} \leqslant -\rho^{(4)}$$

$$\text{or, } (2):C \qquad \frac{1}{d_n} \log \frac{a_{n+1}d_n}{a_n d_{n+1}} \leqslant -\rho.^{(4)*}$$

The divergence criteria (1) : D and (2) : D corresponding to the above, with the inequality signs reversed and $\rho \leqslant 0$, are trivial.

(*B*) Taking $F(x) = \dfrac{1}{x l_1 x l_2 x \ldots (l_p x)^{1+p}}$, where $l_1 x = \log x$, $l_2 x = \log \log x, \ldots$ and x is so large that $l_{p+1} x > 0$, and hence $f(x) = -\dfrac{1}{x} - \dfrac{1}{x l_1 x} - \ldots - \dfrac{1+\rho}{x l_1 x \ldots l_p x}$, we infer that $\sum^{\infty} a_n$ is convergent when $\rho > 0$ and

$$(3):C \qquad \frac{\log \left(\dfrac{a_n D_n l_1 D_n \ldots l_p D_n}{d_n} \right)}{l_{p+1} D_n} \leqslant -\rho^{(4)}$$

$$\text{or, } (4):C \quad \frac{1}{d_n} \log \frac{a_{n+1} d_n}{a_n d_{n+1}} \leqslant -\frac{1}{D_n} - \frac{1}{D_n l_1 D_n} - \ldots - \frac{1+\rho}{D_n l_1 D_n \ldots l_p D_n}.$$

The divergence criteria (3) : D and (4) : D corresponding to the above are obtained by reversing the inequality signs and taking $\rho \leqslant 0$.

(*C*) When $\dfrac{a_n}{d_n} = F(D_n)$, $d_n = 1$, Theorem 1 reduces to the Maclaurin-Cauchy integral test. When $\dfrac{1}{d_n} \log \dfrac{a_{n+1} d_n}{a_n d_{n+1}} = f(D_n)$, $d_n = 1$, Theorem 2 results in the following:

* This includes Kummer's test. *Vide* Ref. (5) p. 412.

If $\sum^{\infty} a_n$ is a series of positive terms and if $r(x)$ is a function such that (i) $r(n) = r_n = \frac{a_{n+1}}{a_n}$, (ii) $0 < \lambda \leqslant r(x) \leqslant \mu$, (iii) $r'(x)$ exists and is continuous, $\int^{\infty} |r'(x)|\,dx$ is convergent, then the convergence of the integral $\int^{\infty} e^{\int^{x} \log r(t)dt}\,dx$ is necessary and sufficient for the convergence of the series $\sum^{\infty} a_n$(1).

2.3. If, in Theorem 2, $\int^{\infty} |f'(x)|\,dx$ is divergent, the inequality (iv) relating $\sum^{\infty} a_n$, $\sum^{\infty} d_n$ to the function f can be modified so that $\sum^{\infty} a_n$, still reflects the behaviour of $\int^{\infty} e^{\int^{x} f(t)dt}\,dx$, provided $f(x)$ has a continuous derivative of some order which gives rise to an absolutely convergent integral at ∞.

Transforming the expression j_ν in 2.1 by a further integration by parts, we obtain

$$\int_{D_{\nu-1}}^{D_\nu} f(t)dt - d_\nu f(D_\nu) + \frac{d_\nu}{2}\int_{D_{\nu-1}}^{D_\nu} f'(t)dt$$

$$= \tfrac{1}{2}\int_{D_{\nu-1}}^{D_\nu} (t-D_{\nu-1})(t-D_\nu)f''(t)dt = j_\nu^{(2)} \text{ (say)}.$$

And generally, if we denote by $\phi_k(x)$ the Bernoullian polynomial of degree k,

$$j_\nu^{(2k+2)} = \int_{D_{\nu-1}}^{D_\nu} f(t)dt - \Big\{d_\nu f(D_\nu) - \frac{d_\nu}{2}\int_{D_{\nu-1}}^{D_\nu} f'(t)dt$$

$$-\frac{B_1}{2!}d_\nu^2\int_{D_{\nu-1}}^{D_\nu} f''(t)dt + \frac{B_2}{4!}d_\nu^4\int_{D_{\nu-1}}^{D_\nu} f^{(4)}(t)dt + \dots$$

$$+\frac{(-1)^k B_k}{(2k)!}d_\nu^{2k}\int_{D_{\nu-1}}^{D_\nu} f^{(2k)}(t)dt\Big\}$$

$$= \frac{-d_\nu^{2k+1}}{(2k+1)!}\int_{D_{\nu-1}}^{D_\nu} \phi_{2k+1}\Big(\frac{t-D_{\nu-1}}{d_\nu}\Big) f^{(2k+1)}(t)dt$$

$$= \frac{d_\nu^{2k+2}}{(2k+2)!}\int_{D_{\nu-1}}^{D_\nu} \phi_{2k+2}\left(\frac{t-D_{\nu-1}}{d_\nu}\right) f^{(2k+2)}(t)dt^*.$$

This leads us immediately to

THEOREM 2-a. *If* (i), (ii) *are as in Theorem* 2, (iii) $f(x)$ *has continuous derivatives of all orders concerned and either* $\int^\infty |f^{(2k+1)}(x)|dx$ *or* $\int^\infty |f^{(2k+2)}(x)|dx$ *is convergent, then* $\sum^\infty a_n$ *is convergent provided*

$$\text{(iv)} \quad \frac{1}{d_n} \log \frac{a_{n+1}d_n}{a_n d_{n+1}} \leqslant \tfrac{1}{2}\left\{ f(D_n)+f(D_{n-1}) \right\}$$

$$-\frac{B_1}{2!}d_n\left\{ f'(D_n)-f'(D_{n-1}) \right\}+\cdots$$

$$+\frac{(-1)^k B_k}{(2k)!}d_n^{2k-1}\left\{ f^{(2k-1)}(D_n)-f^{(2k-1)}(D_{n-1}) \right\},$$

$$\text{(v)} \quad \int^\infty e^{\int^x f(t)dt}\, dx \text{ is convergent.}$$

Also, $\sum^\infty a_n$ *is divergent provided the inequality sign in* (iv) *is reversed and the integral in* (v) *is divergent, other conditions remaining unaltered.*

It will be observed that the relation between Theorems 2 and 2-a is similar to that between Hardy's generalization of the Maclaurin-Cauchy integral test and its further generalization.(2)

3. In the theorems which follow, we dispense with the condition $d_n = O(1)$.

THEOREM 3. *If* (i) $\sum^\infty a_n$ *is a series of positive terms,* (ii) $\sum^\infty d_n$ *is a divergent series of positive terms,* (iii) $F(x)$ *is positive monotone decreasing, then* $\sum^\infty a_n$ *is convergent* [*or divergent*] *provided.*

$$\text{(iv)} \quad \frac{a_n}{d_n} \leqslant F(D_n) \; [\text{or} \geqslant F(D_{n-1})],$$

* The proof of this result is on the lines of that of the "Euler-Maclaurin Sum Formula".

(v) $\int^{\infty} F(x)dx$ *is convergent* [*or divergent*].

PROOF. The theorem issues from one of de la Vallee-Poussin: $\sum^{\infty} d_n F(D_n)$ converges with $\int^{\infty} F(x)dx$ and $\sum^{\infty} d_n F(D_{n-1})$ diverges with $\int^{\infty} F(x)dx$.(7)

COROLLARY. If $\frac{D_n}{D_{n-1}} = O(1)$, we may replace the second alternative of (iv) by $\frac{a_n}{d_n} \geqslant F(D_n)$.

3.1. DEDUCTIONS FROM THEOREM 3. (A) From the Corollary we can derive successively Cauchy's condensation test and Abel's (or Pringsheim's) test for a series of positive decreasing terms.

(B) The Theorem shows that we can state (1) : C and (3) : C without the restriction $(d_n) = O(1)$. The Corollary shows that we can replace the restriction $d_n = O(1)$ in (3) : D by the less stringent one, $\frac{D_n}{D_{n-1}} = O(1)$.

4. THEOREM 4. *If* (i), (ii) *are as in Theorem* 3, *and* (iii) $-f(x)$ *is positive monotone decreasing, then* $\sum^{\infty} a_n$ *is convergent* [*or divergent*] *provided*

(iv) $\frac{1}{d_n} \log \frac{a_n d_{n-1}}{a_{n-1} d_n} \leqslant f(D_{n-1})$

$\left[or,\ \frac{1}{d_n} \log \frac{a_{n+1} d_n}{a_n d_{n+1}} \geqslant f(D_n) \right]$,

(v) $\int^{\infty} e^{\int^x f(t)dt} dx$ *is convergent* [*or divergent*].

PROOF. If the first alternatives of (iv) and (v) are given,

$$\log \frac{a_\nu d_{\nu-1}}{a_{\nu-1} d_\nu} \leqslant \int_{D_{\nu-1}}^{D_\nu} f(t)dt + \int_{D_{\nu-1}}^{D_\nu} \left\{ f(D_{\nu-1}) - f(t) \right\} dt$$

$$\leqslant \int_{D_{\nu-1}}^{D_\nu} f(t)dt.$$

Sum this inequality from $\nu = m+1$ to $\nu = n$ and add the sum to

$$0 \leqslant -\int_x^{D_n} f(t)\,dt \qquad (D_{n-1} \leqslant x \leqslant D_n).$$

Then
$$\frac{a_n}{d_m} \leqslant \frac{a_m}{d_m} e^{\int_{D_m}^{x} f(t)dt} \qquad (D_{n-1} \leqslant x \leqslant D_n).$$

If, on the other hand, the second alternatives of (iv) and (v) are given,

$$\log \frac{a_{\nu+1}d_\nu}{a_\nu d_{\nu+1}} \geqslant \int_{D_{\nu-1}}^{D_\nu} f(t)\,dt + \int_{D_{\nu-1}}^{D_\nu} \{ f(D_\nu) - f(t) \}\,dt$$
$$\geqslant \int_{D_{\nu-1}}^{D_\nu} f(t)\,dt.$$

Sum the last inequality from $\nu = m+1$ to $\nu = n-1$ and add the sum to

$$0 \geqslant \int_{D_{n-1}}^{x} f(t)\,dt \qquad (D_{n-1} \leqslant x \leqslant D_n).$$

Then
$$\frac{a_n}{d_n} \geqslant \frac{a_{m+1}}{d_{m+1}} e^{\int_{D_m}^{x} f(t)dt} \qquad (D_{n-1} \leqslant x \leqslant D_n).$$

In either case, the proof can be completed on the lines already indicated.

4.1. Deductions from Theorem 4. Taking in succession $f(x) = -\rho < 0$,(4) $f(x) = -\frac{1}{x} - \frac{1}{xl_1x} - \ldots - \frac{1+\rho}{xl_1x\ldots l_px}$ $(\rho > 0)$, we observe that the factor $\frac{1}{d_n}$ prefixed to the logarithm in the left hand members of (2) : C and (4) : C, has to be replaced by $\frac{1}{d_{n+1}}$ when $\overline{\lim}\, d_n$ is infinite. The divergence criteria (2) : D and (4) : D are however valid, whether $\overline{\lim}\, d_n$ is finite or infinite.

5. Our theorems show that all the convergence criteria for a series of positive terms, barring those of the Ermakoff group, may be viewed as resulting from a comparison of the given series with a *divergent* series of positive terms. The process involved in such a comparison is best described as 'condensation' by means of an integral. This method of condensation can also be used with advantage to derive properties other than those of convergence or divergence, of series of the type:—

(a) $\sum d_n F(D_{n-1})$ where (i) (D_n) is an increasing divergent sequence and $d_n = D_n - D_{n-1}$, (ii) $F(x)$ is positive monotone decreasing;

(b) $\sum^{\infty} c_n F(C_{n-1})$ where (i) (C_n) is a decreasing null sequence and $c_n = C_{n-1} - C_n$, (ii) $F(x)$ is as in (a).[6]

REFERENCES

(1) BRINK, R. W. A new integral test for the convergence and divergence of infinite series, *Trans. Amer. Math. Soc.*, 19 (1918), 186-204.

(2) HARDY, G. H. Theorems connected with Maclaurin's test for the convergence of series, *Proc. Lond. Math. Soc.* (2), 9 (1911), 126 144.

(3) LITTLEWOOD, J. E. Note on the convergence of series of positive terms, *Messenger of Maths.*, 39 (1910) 191-192.

(4) PRINGSHEIM, A. Allegemeine Theorie der Divergenz und Convergenz von Reihen mit Positiven Gliedern, *Math. Annalen*, 35 (1890), 297-394.

(5) RAJAGOPAL, C. T. On an integral test of R. W. Brink for the convergence of series, *Bull. Amer. Math. Soc.*, 43 (1937), 405-412.

(6) RAJAGOPAL, C. T. On certain theorems of Pringsheim, *Tohoku Math. Journal*, 43 (1937), 122-126.

(7) DE LA VALLEE-POUSSIN. *Cours d'Analyse* 1, (1926), 398.

CORRIGENDA

I. *By* N. BASAVA RAJU

In my paper* "On certain Symmetric Functions of Numbers prime to m", the proof of the statement that $S_r \equiv 0 \pmod{m^2}$ is equivalent to

$$\mathbf{S} a_i^{r-1} \equiv 0 \pmod{m}$$

fails when r is negative, where the symbol $\mathbf{S}$ denotes $\sum\limits_{\substack{i=1 \\ a_i < m/2}}^{h/2}$

The proof for a negative r is as follows:

Let $\qquad r = -u.$

Now $\displaystyle\sum_{i=1}^{h} a_i^r \equiv \mathbf{S}\left\{\frac{1}{a_i^u} + \frac{1}{(m-a_i)^u}\right\} \equiv \mathbf{S}\,\frac{uma_i^{u-1}}{a_i^u(m-a_i)^u} \pmod{m^2}.$

But since $\displaystyle\qquad \frac{a_i^{u-1}}{a_i^u(m-a_i)^u} + \frac{1}{a_i^{u+1}} \equiv 0 \pmod{m},$

the statement that

$$\mathbf{S}\frac{a_i^{u-1}}{a_i^u(m-a_i)^u} \equiv 0 \pmod{m}$$

is equivalent to

$$\mathbf{S}\frac{1}{a_i^{u+1}} \equiv 0 \pmod{m}, \text{ or to } \mathbf{S}a^{r-1} \equiv 0 \pmod{m}.$$

This is again equivalent to $S_{r-1} \equiv 0 \pmod{m}$, because if l be such that $lh+r-1 > 0$,

$$\mathbf{S}a_i^{r-1} \equiv 0 \pmod{m} \text{ is the same as } \mathbf{S}a_i^{lh+r-1} \equiv 0 \pmod{m},$$

or $\qquad \mathbf{S}a_i^{lh+r-1} \equiv 0 \pmod{m}$, i.e. $S_{r-1} \equiv (0 \bmod m)$.

II. *By* R. S. VARMA.

In my recent paper in the *Jour. Ind. Math. Soc.* (2) 3 (1938) 25-33, the following corrections are necessary.

Page 26 line 18: *for* 'true when' *read* 'true even when'.

" 31 after line 20, add

"and

$$\int_0^{\infty} x^{n+\frac{1}{2}} e^{-\frac{1}{4}x^2} D_{2n-2}(x) J_{n-\frac{1}{2}}(ax)\,dx$$
$$= (-)^{n-1}\Gamma(n+1)2^n a^{n-\frac{5}{2}} e^{-\frac{1}{4}a^2} K_{2n}(\tfrac{1}{4}a^2)."$$

* *Jour. Indian Math. Soc.* (New Series) 2, 310.

A TAUBERIAN THEOREM ON (λ, k)-PROCESS OF SUMMATION

By S. MINAKSHISUNDARAM, *Madras University.*

[Received 1 May 1938]

1. I propose to prove here a generalization of a theorem due to Karamata* on the (λ, k)-process of summation. This result includes all known Tauberian theorems on this process and can be applied to prove similar results on power series and Dirichlet's series.

1.1. Let $A^{(k)}(\omega)$ be the kth Riesz's sum of the series $\sum_{\nu=1}^{\infty} a_\nu$ with respect to the sequence $[\lambda_n]$, $0 < \lambda_1 < \lambda_2 \ldots < \lambda_n \to \infty$. Let

$$R_k(\omega) = \frac{A^{(k)}(\omega)}{\omega^k} = \sum_{\lambda_n \leqslant \omega} \left(1 - \frac{\lambda_n}{\omega}\right)^k a_n$$

$$= \frac{k}{\omega^k} \int_0^\omega (\omega - t)^{k-1} A(t)\, dt,$$

where $A(t) = A^{(0)}(t)$. We write

$$\overline{\lim_{\omega\to\infty}}\, R_k(\omega) = u_k, \quad \underline{\lim_{\omega\to\infty}}\, R_k(\omega) = l_k$$

and $u = u_0$, $l = l_0$. The following theorem constitutes the main result of this paper.

THEOREM 1. *Let*

$$\underline{\lim_{n\to\infty}} \frac{\lambda_n a_n}{\lambda_n - \lambda_{n-1}} = -\alpha \leqslant 0. \tag{1}$$

Then

(i) *if* $k > 0$,

$$\left.\begin{aligned} u_k &\leqslant u_{k+1} + 2\sqrt{\alpha(u_{k+1} - l_{k+1})} \\ l_k &\geqslant l_{k+1} - 2\sqrt{\alpha(u_{k+1} - l_{k+1})} \end{aligned}\right\}; \tag{2}$$

(ii) *if* $k = 0$,

$$\left.\begin{aligned} u &\leqslant u_1 + 2\sqrt{\alpha(u_1 - l_1)} \\ l &\geqslant l_1 - \alpha \end{aligned}\right\}; \tag{3}$$

* *Proc. Lond. Math. Soc.* 43 (1937), 20-25.

(iii) *if* $\frac{\lambda_n}{\lambda_{n-1}} \to 1$, $l \geqslant l_1 - 2\sqrt{\alpha(u_1 - l_1)}$ *in case* (ii).

2. PROOF OF THEOREM 1. Firstly we note that

$$l_{k+1} \leqslant \varliminf_{\omega\to\infty} \frac{1}{\omega}\int_0^\omega R_k(t)\,dt \leqslant u_{k+1}, \tag{4}$$

for we have

$$\int_0^\omega R_k(t)\,dt = \int_0^\omega \frac{A^{(k)}(t)}{t^k}\,dt$$

$$= \left[\frac{A^{(k+1)}(\omega)}{(k+1)\omega^k}\right] + \frac{k}{k+1}\int_0^\omega R_{k+1}(t)\,dt \tag{5}$$

on integration by parts and using the relation

$$\frac{d}{d\omega}\,\frac{A^{(k+1)}(\omega)}{k+1} = k\int_0^\omega (\omega - t)^{k-1} A(t)\,dt = A^{(k)}(\omega).$$

From (5) we easily deduce (4), since

$$\frac{A^{k+1}(\omega)}{\omega^{k+1}} = R_{k+1}(\omega).$$

Next if we write $b_n = a_n\lambda_n$, and if $B^{(k)}(\omega)$ be the kth Riesz's sum for Σb_n, it can be verified* that, for $k > 0$,

$$R_k(\omega) = k\int_0^\omega \frac{B^{(k-1)}(t)}{t^{k+1}}\,dt. \tag{6}$$

Using (1) and (6) we get, if $\delta > 0$,

$$\varliminf_{\substack{\omega\to\infty \\ \omega\leqslant x\leqslant\omega(1+\delta)}} [R_k(x) - R_k(\omega)] = \varliminf_{\omega\to\infty} k\int_\omega^x \frac{B^{(k-1)}(t)}{t^{k+1}}\,dt, \tag{7}$$

while

$$\varliminf_{\omega\to\infty} \frac{B^{(k-1)}(\omega)}{\omega^k} = \varliminf_{\omega\to\infty} \frac{\Sigma(\omega - \lambda_n)^{k-1} b_n}{\omega^k}$$

$$\geqslant \frac{-\alpha\int_0^\omega (\omega - t)^{k-1}\,dt}{\omega^k} \geqslant -\frac{\alpha}{k}. \tag{8}$$

Combining (7) and (8), we find that

$$\varliminf_{\substack{\omega\to\infty \\ \omega\leqslant x\leqslant\omega(1+\delta)}} [R_k(x) - R_k(\omega)] \geqslant -\alpha\log(1+\delta) \geqslant -\alpha\delta. \tag{9}$$

* Hardy and Riesz, *Dirichlet's Series* (Cambridge Tracts, No. 18).

We can now complete the proof for the case $k > 0$. We have

$$\int_0^{\omega(1+\delta)} R_k(t)\,dt - \int_0^{\omega} R_k(t)\,dt$$

$$= \int_{\omega}^{\omega(1+\delta)} R_k(t)dt = \omega\delta\, R_k(\omega) + \int_{\omega}^{\omega(1+\delta)} [R_k(t) - R_k(\omega)]dt$$

from which we get

$$\delta\omega R_k(\omega) = \int_0^{\omega(1+\delta)} R_k(t)dt - \int_0^{\omega} R_k(t)dt - \int_{\omega}^{\omega(1+\delta)} [R_k(t) - R_k(\omega)]dt. \quad (10)$$

Using (4) and (9) in (10), we conclude that

$$u_k \leqslant \frac{1+\delta}{\delta} u_{k+1} - \frac{1}{\delta} l_{k+1} + \alpha\delta. \quad (11)$$

Since (11) is true for all $\delta > 0$, we can choose that value of δ which makes the right side of (11) a minimum and thus arrive at the first relation in (2). Similarly starting from the identity

$$\delta w R_k[\omega(1+\delta)] = \int_0^{\omega(1+\delta)} R_k(t)dt - \int_0^{\omega} R_k(t)dt$$

$$+ \int_{\omega}^{\omega(1+\delta)} [R_k\{\omega(1+\delta)\} - R_k(t)]dt \quad (12)$$

we can obtain the second relation in (2). When $k = 0$, the function $A(\omega) = A^0(\omega)$ is discontinuous at the points $\omega = \lambda_n$. In this case the relation (9) does not always hold but it is easily seen to hold when $\omega = \lambda_n \to \infty$. The formula (10) holds, and noting that

$$\int_0^{\omega} A(t)\,dt = A^{(1)}(\omega)$$

we can obtain from (10), with $\omega = \lambda_n$, the relation

$$u \leqslant \frac{1+\delta}{\delta} u_1 - \frac{l_1}{\delta} + \alpha\delta$$

from which the first relation in (3) follows as before. The formula (12) also holds, but (9) does not hold for $\omega = \lambda_n(1+\delta)$, so that we cannot use it to obtain a similar relation for l. But from the identity

$$A(\omega) = \frac{A^{(1)}(\omega) + B(\omega)}{\omega}$$

we can conclude, using (1), that

$$l \geqslant l_1 - \alpha.$$

But in case $\frac{\lambda_n}{\lambda_{n-1}} \to 1$ it can be easily shown that (9) holds for all ω even when $k = 0$, so that in this case we can use (12) to get the relation

$$l \geqslant l_1 - 2\sqrt{\alpha(u_1 - l_1)}.$$

2.1. From Theorem 1 we deduce that *if*

$$\varlimsup_{n\to\infty} \frac{\lambda_n a_n}{\lambda_n - \lambda_{n-1}} = -\alpha = 0$$

then

$$\operatorname*{Osc.}_{\omega\to\infty} A(\omega) = \operatorname*{Osc.}_{\omega\to\infty} R_k(\omega)$$

for all $k > 0$. Also under the hypothesis that α is finite in Theorem 1, $R_k(\omega)$ is bounded for all $k > 0$ if it be bounded for some k, and $R_k(\omega)$ tends to a definite limit for all $k > 0$ if it does so for some k. That is, Σa_n is bounded (λ, k) for all $k > 0$ or is never so; and Σa_n is summable (λ, k) for all $k > 0$ or is never summable. The further assumption $\lambda_n/\lambda_{n-1} \to 1$ ensures the same conclusions for all $k \geqslant 0$.

2.2. Similarly we can prove the following result.

THEOREM 2. *Let* $A(t)$ *be a function defined in* $(0, \infty)$ *and of bounded variation in every finite interval. Let* $A^{(k)}(\omega)$, u_k *and* l_k *be defined by the relations at the commencement of* § 1.1. *Let*

$$\varlimsup_{\omega\to\infty} [A(u\omega) - A(\omega)] = -\alpha(u),\ u > 0.$$

Let $\lambda > 1$. *Then*

$$\left.\begin{aligned} u_k &\leqslant u_{k+1} + \frac{u_{k+1} - l_{k+1}}{\lambda - 1} + \frac{1}{\lambda - 1}\int_1^{\lambda} \alpha(t)\,dt \\ l_k &\geqslant l_{k+1} - \frac{u_{k+1} - l_{k+1}}{\lambda - 1} - \frac{1}{\lambda - 1}\int_{\frac{1}{\lambda}}^{1} \alpha(t)\,dt \end{aligned}\right\}$$

ON V. RAMASWAMI'S TAUBERIAN THEOREM ON OSCILLATION

By S. MINAKSHISUNDARAM, *Madras University.*

[Received 1 July 1938]

1. Let $f(x)$ be the power series

$$f(x) = \sum_{\nu=0}^{\infty} a_\nu x^\nu \tag{1}$$

and let its radius of convergence be unity. Let

$$s_n = a_0 + a_1 + \ldots + a_n. \tag{2}$$

V. Ramaswami* has proved the following result:

THEOREM A. *If*

$$\lim_{\delta \to 0} \varliminf_{\substack{n \to \infty \\ n \leqslant m \leqslant n(1+\delta)}} \frac{1}{\delta}(s_m - s_n) \geqslant 0 \tag{3}$$

and $f(x) = O(1)$ *as* $x \to 1-0$, *then*†

$$\underset{n \to \infty}{\text{Osc.}} (s_n) = \underset{x \to 1-0}{\text{Osc.}} f(x).$$

1.1. In this paper I give a simple and direct proof of Theorem A based on certain auxiliary theorems interesting in themselves. The method of proof closely resembles the one used by Szász in some of his recent papers.‡

2. We shall deduce Theorem A from two auxiliary theorems.

Let

$$U = \varlimsup_{x \to 1} f(x), \quad L = \varliminf_{x \to 1} f(x)$$

and

$$u = \varlimsup_{n \to \infty} s_n, \qquad l = \varliminf_{n \to \infty} s_n.$$

We first prove the following two results:

* *Jour. Lond. Math. Soc.* 10 (1935), 294-308.

† Osc. means oscillation.

‡ *Duke Math. Journal* 1 (1935), 105-111. *Trans. American Math. Soc.* 39 (1936), 117-130.

THEOREM 1. *If U and L are finite and*

$$\overline{\lim_{n\to\infty}}\, na_n \leqslant \beta, \quad \underline{\lim_{n\to\infty}}\, na_n \geqslant -\alpha \tag{4}$$

where $0 \leqslant \alpha, \beta < \infty$, *then*

$$\left.\begin{aligned} u &\leqslant U+2\sqrt{\alpha\beta} \\ l &\geqslant L-2\sqrt{\alpha\beta} \end{aligned}\right\}. \tag{5}$$

THEOREM 2. *If* $\lim_{n\to\infty} na_n = 0$ *and* $f(x) = O(1)$, *then*

$$\underset{n\to\infty}{\text{Osc.}}\, s_n = \underset{x\to 1}{\text{Osc.}}\, f(x).$$

2.1. PROOF OF THEOREM 1. We have

$$\begin{aligned} s_n - f(x) &= \sum_{\nu=1}^{n} a_\nu(1-x^\nu) - \sum_{\nu=n+1}^{\infty} a_\nu x^\nu \\ &= S_n^{(1)} - S_n^{(2)}, \text{ say.} \end{aligned} \tag{6}$$

Let $\delta > 0$ and take $x = 1-\delta/n$ in (6). We get

$$\begin{aligned} \overline{\lim}\, S_n^{(1)} &\leqslant \overline{\lim_{n\to\infty}}\, \frac{\delta}{n} \sum_{\nu=1}^{n} \nu a_\nu \frac{1-x^\nu}{(1-x).\nu} \\ &\leqslant \overline{\lim_{n\to\infty}}\, \frac{\delta}{n} \sum_{\nu=1}^{n} \beta < \beta\delta \end{aligned} \tag{7}$$

and

$$\underline{\lim}\, S_n^{(1)} \geqslant -\alpha\delta \tag{8}$$

in a similar manner. Also,

$$\begin{aligned} \overline{\lim_{n\to\infty}}\, S_n^{(2)} &\leqslant \overline{\lim_{n\to\infty}} \sum_{n+1}^{\infty} \nu a_\nu . \frac{x^\nu}{\nu} \\ &\leqslant \overline{\lim_{n\to\infty}}\, \frac{\beta x^{n+1}}{(n+1)(1-x)} \\ &\leqslant \frac{\beta}{\delta}\, \overline{\lim_{n\to\infty}} \left(1-\frac{\delta}{n}\right)^{n+1} < \frac{\beta e^{-\delta}}{\delta} \end{aligned} \tag{9}$$

and similarly

$$\underline{\lim_{n\to\infty}}\, S_n^{(2)} \geqslant -\frac{\alpha e^{-\delta}}{\delta} \tag{10}$$

Combining the relations (6)-(10) and noting that $e^{-\delta} \leqslant 1$ for $\delta \geqslant 0$, we get

$$\left.\begin{aligned} u &\leqslant U+\beta\delta+\frac{\alpha}{\delta} \\ l &\geqslant L-\alpha\delta-\frac{\beta}{\delta} \end{aligned}\right\} \tag{11}$$

from which we get (5) by taking the respective values of δ which make the right sides of (11) minimum.

2.2. Proof of Theorem 2. Write

$$b_n = \frac{\sum_0^n \nu a_\nu}{n+1} \tag{12}$$

and

$$f_1(x) = \sum_0^\infty \frac{b_n}{n} x^n \tag{13}$$

If

$$\sigma_n = \frac{s_0+s_1+\ldots+s_n}{n+1},$$

it is easy to see that

$$f_1(x) = (1-x)\sum_0^\infty \sigma_n x^n \tag{14}$$

from whch it is easily deduced that

$$\text{Osc. } f_1(x) \leqslant \text{Osc. } f(x). \tag{15}$$

Since

$$\underline{\lim}_{n\to\infty}\ n a_n = 0 \tag{16}$$

it follows that

$$\underline{\lim}_{n\to\infty}\ b_n \geqslant 0. \tag{17}$$

If we *also prove* that

$$\overline{\lim}_{n\to\infty}\ b_n < \infty, \tag{18}$$

it would follow from Theorem 1, and the relations (14)-(18) that

$$\begin{aligned} \underset{n\to\infty}{\text{Osc.}}\ \sigma_n &= \underset{x\to 1}{\text{Osc.}}\ f_1(x) \\ &\leqslant \underset{x\to 1}{\text{Osc.}}\ f(x) \\ &\leqslant \underset{n\to\infty}{\text{Osc.}}\ s_n \end{aligned} \tag{19}$$

the last inequality being always true without any hypothesis on a_n. By a method similar to the one I have used elsewhere*, it can be shown that

$$\text{Osc. } \sigma_n = \text{Osc. } s_n \tag{20}$$

when (16) holds, so that (19) and (20) give the result required. It is therefore sufficient to prove (18) when (16) holds. Now (16) implies that

$$\underline{\lim}_{x\to 1}(1-x)f'(x) \geqslant 0 \tag{21}$$

and

$$\underline{\lim}_{x\to 1}(1-x)^2 f''(x) = \underline{\lim}_{x\to 1}(1-x)^2 \sum n(n-1)a_n x^{n-2}.$$
$$\geqslant 0. \tag{22}$$

By Taylor's theorem, if $0 < \delta < 1$,

$$f(x+\delta.\overline{1-x}) = f(x)+\delta(1-x)f'(x)+\frac{\delta^2(1-x)^2}{2}f''(\xi), \tag{23}$$

where $x \leqslant \xi \leqslant x+\delta(1-x)$. From (23) we get

$$(1-x)f'(x) = \frac{f(x+\delta.\overline{1-x})-f(x)}{\delta} - \frac{\delta(1-x)^2}{(1-\xi)^2}(1-\xi)^2 f''(\xi),$$

from which we conclude, in virtue of (22), that

$$\overline{\lim}_{x\to 1}(1-x)f'(x) \leqslant \frac{U-L}{\delta}. \tag{24}$$

Letting $\delta \to 1$ in (24), we get

$$\overline{\lim}_{x\to 1}(1-x)f'(x) \leqslant U-L,$$

from which in conjunction with (21) we obtain

$$(1-x)f'(x) = O(1) \tag{25}$$

as $x\to 1$. Now, by a known result due to Hardy and Littlewood† we find that (16) and (25) imply the relation

$$\overline{\lim}_{n\to\infty} b_n \leqslant e\, \overline{\lim}_{x\to 1}(1-x)f'(x)$$
$$\leqslant e(U-L)$$

which is equivalent to the relation (18) since $f(x) = O(1)$ as $x \to 1$. So Theorem 2 is proved.

* "A Tauberian theorem on (λ, k) process of summation", *Journ. Indian Math. Soc.* 3 (1938), 127-130.

† *Proc. London Math. Soc.* (2) 13 (1914), 174-191.

3. Theorem A can now be deduced from Theorem 2. In the notation of § 2.2 we have

$$b_n = \frac{\sum_0^n \nu a_\nu}{n+1} = \frac{s_n - s_0 + s_n - s_1 + \ldots + s_n - s_{n-1}}{n+1}. \tag{26}$$

Using (26) and the condition (3), it can be shown* that

$$\varliminf_{n\to\infty} b_n \geqslant 0. \tag{27}$$

From (14) we get

$$f_1(x) = (1-x)\int_0^x \frac{f(t)}{(1-t)^2}\,dt = O(1), \tag{28}$$

since by hypothesis $f(x) = O(1)$ as $x \to 1$. Hence applying Theorem 2 to the series (14), we get in virtue of (13), (27) and (28)

$$\text{Osc. } \sigma_n = \text{Osc. } f_1(x)$$

from which we deduce as in (19),

$$\text{Osc. } \sigma_n \leqslant \text{Osc. } f(x) \leqslant \text{Osc. } s_n. \tag{29}$$

But it is known that under hypothesis (3),

$$\text{Osc. } \sigma_n = \text{Osc. } s_n$$

from which, in conjunction with (29), we get Theorem A.

* The detailed proof is a bit long but not difficult. So it is not included in the body of this paper. In fact it consists in showing that

(i) if $c_n = (n+1)b_n$, (3) implies $\lim_{\delta\to 0} \varliminf_{n\to\infty} \frac{1}{\delta}\left(\frac{c_m - c_n}{m}\right) \geqslant 0$

(ii) $\varlimsup_{n\to\infty} b_n = -\Lambda > -\infty$, and (iii) $\Lambda \leqslant 0$.

WARING'S THEOREM FOR POWERS OF PRIMES

By HANSRAJ GUPTA, *Government College, Hoshiarpur.*

[Received 3 June 1938]

1. The ideal Waring's Theorem states that every positive integer is a sum of I integral nth powers $\geqslant 0$, where

$$I = [(\tfrac{3}{2})^n] + 2^n - 2.$$

Chowla[1] and the author[2] have considered the corresponding problem for primes so far as the squares and cubes are concerned. It has been shown that all positive integers $i \leqslant 240{,}000$ are sums of at the most eight squares of primes $p \geqslant 1$. It has also been shown that all positive integers $i \leqslant 100{,}000$; $i \neq 1301$ can be expressed as the sums of at the most twelve cubes of primes; while 1301 requires 13 and no less. In fact it is noticed that integers greater than 8161 require 11 cubes of primes at the most.

I think that for every $n \geqslant n_0 > 6$, where n_0 remains to be determined, every positive integer is a sum of I nth powers of primes at the most.

This conjecture cannot be improved upon in so far as

$$L = [(\tfrac{3}{2})^n]\, 2^n - 1,$$

cannot be expressed as the sum of less than I nth powers of primes. It further appears that for integers $> L$, I_1 powers suffice, where $I_1 < I$. Beyond the last one requiring I_1, all integers are sums of at most $I_2 < I_1$ powers, and so on. Thus Professor Dickson's[3] remarks seem to apply to powers of primes as well.

In what follows, decompositions into 12th powers of primes are studied in some detail.

2. Let

$$\begin{aligned}
a_1 &= 2^{12} = 4\ 096,\\
a_2 &= 3^{12} = 531\ 441,\\
a_3 &= 5^{12} = 244\ 140\ 625,\\
a_4 &= 7^{12} = 13\ 841\ 287\ 201,\\
a_5 &= 11^{12} = 3\ 138\ 428\ 376\ 721,\\
a_6 &= 13^{12} = 23\ 298\ 085\ 122\ 481,
\end{aligned}$$

and so on.

Any given number $N < a_{r+1}$ can be expressed as

$$A_r a_r + A_{r-1} a_{r-1} + \ldots + A_2 a_2 + A_1 a_1 + A_0,$$

where $A_0 < a_1$, $A_1 a_1 + A_0 < a_2$, $A_2 a_2 + A_1 a_1 + A_0 < a_3$, and so on. This way of expression of N is unique. The sum ΣA is called the absolute weight of N, and is written $Ab.\ (N)$.

The number N can be expressed as a linear homogeneous function of $1, a_1, a_2, a_3, \ldots, a_r$, with integral coefficients $\geqslant 0$, in many other ways. Let one such way be:

$$N = B_r a_r + B_{r-1} a_{r-1} + \ldots + B_3 a_3 + B_2 a_2 + B_1 a_1 + B_0.$$

Then ΣB is called in general the weight of N. The least value of ΣB is called the minimum weight of N and is written $Min.\ (N)$.

If of all integers between i and j, W has the greatest minimum weight, we call W the heaviest integer in the range. We write $W(i, j)$ to denote the heaviest integer between i and j, and $M(i, j)$ to denote its minimum weight. For brevity, we write $W(i, j)k$ for $W(ik, jk)$ and $M(i, j)k$ for $M(ik, jk)$. In the earlier part of this paper we study the behaviour of $M(i, j)$ in suitably selected ranges.

As we proceed, the interval 0 to 4095, in which the values of A_0 lie, gets broken up into several sub-intervals. We give the inferior limits of these sub-intervals, and state the value Δ by which the absolute weight must be diminished to obtain the minimum weight in any particular case.

3. Our method of obtaining the minimum weights of integers $N > a_r$ mainly consists in expressing N in the following forms:

$$N = (A_r - g)\ a_r + N_g, \qquad g = 0, 1, 2, 3, \ldots, A_r.$$

Evidently $Min.\ (N) = A_r - g + Min.\ (N_g)$, for some g.
To find $Min.\ (N_g)$ we make use of the table of sub-intervals for numbers greater than a_{r-1}. This leads to a new table of sub-intervals for numbers greater than a_r. An upper limit for the values of g is easily obtained by expressing each multiple of a_r in succession in the following manner:

$$q a_r = C_{r-1} a_{r-1} - C_{r-2} a_{r-2} - \ldots - C_1 a_1 - C_0, \qquad q = 1, 2\ 3, \ldots,$$

where $C_0 < a_1$, $C_1 a_1 + C_0 < a_2$, $C_2 a_2 + C_1 a_1 + C_0 < a_3$, etc.

If $C_{r-1} \geqslant q + C_0 + C_1 + C_2 + \ldots + C_{r-2}$ for the first time when $q = q_r$, then the required upper limit for g is $q_r - 1$.

If $\quad [a_{r+1}/a_r] = Q_r > q_r$, and $M(q_r - 1, q_r) a_r = u$;
then $\quad M(q_r + t - 1, q_r + t) a_r = u + t$, $t = 1, 2, \ldots, Q_r - q_r$.

4. We now proceed to consider the forms and the minimum weights of the heaviest integers in suitably selected ranges.

We have $W(1, a_2) = 128a_1+4095$; therefore $M(1, a_2)=4223$.

Now
$$a_2 = 130a_1-1039,$$
$$2a_2 = 260a_1-2078,$$
$$3a_2 = 390a_1-3117,$$
but
$$4a_2 = 519a_1-60.$$

Hence we have the following table of sub-intervals for numbers $> a_2$:

A_2	A_0	Δ
1	1039	910
2	2078	1820
3	3117	2730.

Therefore
$$W(1, 2)a_2 = a_2+128a_1+4095, \quad M(1, 2)a_2 = 3314;$$
$$W(2, 3)a_2 = 2a_2+128a_1+4095, \quad M(2, 3)a_2 = 2405;$$
$$W(3, 4)a_2 = 3a_2+128a_1+4095, \quad M(3, 4)a_2 = 1496.$$

Finally $M(458a_2, a_3) = 1951$.

5. Next we have
$$a_3 = 459a_2+51a_1+310$$
$$= 459a_2+52a_1-3786$$
$$= 460a_2-78a_1-2747;$$
$$2a_3 = 918a_2+102a_1+620$$
$$= 918a_2+103a_1-3476$$
$$= 919a_2-27a_1-2437;$$
while
$$3a_3 = 1379a_2-106a_1-1088.$$

These equations lead to the following table of sub-intervals:

A_3	A_2	A_1	A_0	Δ
1	..	..	729	91
..	1	..	1039	910
2	..	..	1458	182
1	..	..	1768	1001
..	2	..	2078	1820
2	..	27	2437	1547
2	..	..	2497	1092
1	..	78	2747	2366
1	..	..	2807	1911
..	3	..	3117	2730
2	..	..	3476	2457
1	..	..	3786	3276

Considering each sub-interval separately, we get

$$W(1,2)a_3 = a_3+2a_2+77a_1+3785, \quad M(1,2)a_3 = 1954;$$
$$W(2,3)a_3 = 2a_3+a_2+26a_1+2806, \quad M(2,3)a_3 = 1743.$$

Finally $M(56a_3, a_4) = 1797$.

6. We next require the following equations:

$$\begin{aligned}
a_4 &= 56a_3+318a_2+101a_1+267\\
&= 56a_3+318a_2+102a_1-3829\\
&= 56a_3+319a_2-28a_1-2790\\
&= 57a_3-141a_2+50a_1-43\\
&= 57a_3-140a_2-79a_1-3100;\\
2a_4 &= 113a_3+178a_2+21a_1+1263\\
&= 113a_3+178a_2+22a_1-2833\\
&= 113a_3+179a_2-108a_1-1794\\
&= 114a_3-282a_2+100a_1-86\\
&= 114a_3-281a_2-29a_1-3143;\\
3a_4 &= 170a_3+37a_2+71a_1+1220\\
&= 170a_3+37a_2+72a_1-2876\\
&= 170a_3+38a_2-58a_1-1837\\
&= 171a_3-422a_2+21a_1-3186\\
&= 171a_3-421a_2-109a_1-2147;\\
4a_4 &= 226a_3+356a_2+43a_1-1570\\
&= 226a_3+357a_2-87a_1-531\\
&= 227a_3-104a_2+122a_1-2919\\
&= 227a_3-103a_2-8a_1-1880;\\
5a_4 &= 283a_3+215a_2+93a_1-1613\\
&= 283a_3+216a_2-37a_1-574\\
&= 284a_3-244a_2+42a_1-1923\\
&= 284a_3-243a_2-88a_1-884;\\
6a_4 &= 340a_3+75a_2+13a_1-617\\
&= 340a_3+76a_2-116a_1-3674\\
&= 341a_3-385a_2+92a_1-1966\\
&= 341a_3-384a_2-38a_1-927;\\
7a_4 &= 396a_3+393a_2+114a_1-350\\
&= 396a_3+394a_2-15a_1-3407\\
&= 397a_3-66a_2+63a_1-660\\
&= 397a_3-65a_2-66a_1-3717;\\
8a_4 &= 453a_3+253a_2+34a_1+646\\
&= 453a_3+253a_2+35a_1-3450\\
&= 453a_3+254a_2-95a_1-2411\\
&= 454a_3-207a_2+113a_1-703\\
&= 454a_3-206a_2-16a_1-3760;
\end{aligned}$$

$$\begin{aligned}
9a_4 &= 510a_3+112a_2+\ 84a_1+\ 603\\
&= 510a_3+112a_2+\ 85a_1-3493\\
&= 510a_3+113a_2-\ 45a_1-2454\\
&= 511a_3-347a_2+\ 34a_1-3803\\
&= 511a_3-346a_2-\ 96a_1-2764;
\end{aligned}$$

$$\begin{aligned}
10a_4 &= 567a_3-\ 28a_2+\ \ 5a_1-2497\\
&= 567a_3-\ 27a_2-125a_1-1458;
\end{aligned}$$

$$\begin{aligned}
11a_4 &= 624a_3-169a_2+\ 55a_1-2540\\
&= 624a_3-168a_2-\ 75a_1-1501;
\end{aligned}$$

$$\begin{aligned}
12a_4 &= 680a_3+150a_2+\ 26a_1-1234\\
&= 681a_3-310a_2+105a_1-2583\\
&= 681a_3-309a_2-\ 25a_1-1544;
\end{aligned}$$

$$\begin{aligned}
13a_4 &= 737a_3+\ \ 9a_2+\ 76a_1-1277\\
&= 738a_3-450a_2+\ 25a_1-1587\\
&= 738a_3-449a_2-105a_0-\ 548;
\end{aligned}$$

while

$$14a_4 = 794a_3-131a_2-\ \ 4a_1-\ 281.$$

These equations lead to the following table of sub-intervals:

A_4	A_3	A_2	A_1	A_0	Δ
1	..	141	..	43	78
2	..	282	..	86	156
3	..	423	..	129	234
4	..	..	87	531	39
13	..	449	105	548	377
5	..	..	37	574	117
6	..	..	..	617	195
7	..	66	..	660	273
8	..	207	..	703	351
..	1	..	..	729	91
9	..	348	..	746	429
1	..	..	..	772	169
2	..	..	..	815	247
3	..	..	..	858	325
5	..	243	88	884	936
4	..	106	..	901	403
6	..	384	38	927	1014
5	..	247	..	944	481
6	..	388	..	987	559
..	..	1	..	1039	910

A_4	A_3	A_2	A_1	A_0	Δ
1	..	142	..	1082	988
2	..	283	..	1125	1066
3	..	424	..	1168	1144
12	..	..	..	1234	390
13	..	..	..	1277	468
7	..	..	..	1389	364
8	..	..	..	1432	442
1	..	..	107	1441	715
..	2	..	..	1458	182
10	..	27	125	1458	1053
9	..	..	..	1475	520
2	..	..	57	1484	793
1	..	..	..	1501	260
11	..	168	75	1501	1131
3	..	..	7	1527	871
2	..	..	..	1544	338
12	..	309	25	1544	1209
4	..	..	..	1570	949
3	..	..	..	1587	416
13	..	450	..	1587	1287
5	..	..	..	1613	1027
6	..	..	..	1656	1105
7	..	67	..	1699	1183
8	..	208	..	1742	1261
..	1	..	..	1768	1001
9	..	349	..	1785	1339
2	..	..	108	1794	1612
1	..	..	..	1811	1079
3	..	..	58	1837	1690
2	..	..	..	1854	1157
4	..	103	8	1880	1768
3	..	..	..	1897	1235
5	..	244	..	1923	1846
4	..	107	..	1940	1313
6	..	385	..	1966	1924
..	..	2	..	2078	1820
1	..	143	..	2121	1898
3	..	421	109	2147	2509
2	..	284	..	2164	1976
3	..	425	..	2207	2054

A_4	A_3	A_2	A_1	A_0	Δ
12	..	..	..	2273	1300
13	..	..	..	2316	1378
8	..	..	95	2411	1807
7	..	..	..	2428	1274
..	2	..	27	2437	1547
9	..	..	45	2454	1885
8	..	..	..	2471	1352
1	..	..	..	2480	1625
10	..	28	..	2497	1963
2	..	..	..	2523	1703
11	..	169	..	2540	2041
3	..	..	..	2566	1781
12	..	310	..	2583	2119
4	..	..	..	2609	1859
13	..	451	..	2626	2197
5	..	..	..	2652	1937
6	..	..	..	2695	2015
7	..	68	..	2738	2093
..	1	..	78	2747	2366
9	..	346	96	2764	2704
8	..	209	..	2781	2171
1	..	..	28	2790	2444
..	1	..	..	2807	1911
9	..	350	..	2824	2249
2	..	..	..	2833	2522
1	..	..	..	2850	1989
3	..	..	..	2876	2600
4	..	104	..	2919	2678
5	..	245	..	2962	2756
6	..	386	..	3005	2834
1	..	140	79	3100	3263
..	..	3	..	3117	2730
2	..	281	29	3143	3341
1	..	144	..	3160	2808
3	..	422	..	3186	3419
2	..	285	..	3203	2886
7	..	..	15	3407	2639
8	..	..	..	3450	2717
..	2	..	..	3476	2457
9	..	..	..	3493	2795

A_4	A_3	A_2	A_1	A_0	Δ
1	..	..	..	3519	2535
10	..	29	..	3536	2873
2	..	..	..	3562	2613
11	..	170	..	3579	2951
3	..	..	..	3605	2691
12	..	311	..	3622	3029
4	..	..	..	3648	2769
6	..	..	116	3674	3380
5	..	..	..	3691	2847
7	..	65	66	3717	3458
6	..	..	..	3734	2925
8	..	206	16	3760	3536
7	..	69	..	3777	3003
..	1	..	..	3786	3276
9	..	347	..	3803	3614
8	..	210	..	3820	3081
1	..	..	..	3829	3354
2	..	..	..	3872	3432
3	..	..	..	3915	3510
4	..	105	..	3958	3588
5	..	246	..	4001	3666
6	..	387	..	4044	3744

Hence we obtain the following results:

$W(1,2)a_4 = a_4+56a_3+106a_1+1767,\quad M(1,2)a_4=1670;$

$W(2,3)a_4 = 2a_4+56a_3+56a_1+1767,\quad M(2,3)a_4=1543;$

$W(3,4)a_4 = 3a_4+56a_3+6a_1+1767,\quad M(3,4)a_4=1416;$

$W(4,5)a_4 = 4a_4+55a_3+421a_2+28a_1+3785,\quad M(4,5)a_4=1407;$

$W(5,6)a_4 = 5a_4+55a_3+421a_2+28a_1+3785,\quad M(5,6)a_4=1408;$

$W(6,7)a_4 = 6a_4+55a_3+421a_2+28a_1+3785,\quad M(6,7)a_4=1370;$

$W(7,8)a_4 = 7a_4+55a_3+421a_2+28a_1+3776,\quad M(7,8)a_4=1362;$

$W(8,9)a_4 = 8a_4+55a_3+421a_2+28a_1+3733,\quad M(8,9)a_4=1359;$

$W(9,10)a_4 = 9a_4+55a_3+421a_2+28a_1+3733,\quad M(9,10)a_4=1360;$

$W(10,11)a_4 = 10a_4+55a_3+421a_2+28a_1+3733, M(10,11)a_4=1361;$

$W(11,12)a_4 = 11a_4+55a_3+421a_2+15a_1+3776, M(11,12)a_4=1327;$

$W(12,13)a_4 = 12a_4+56a_3+a_2+26a_1+2470, M(12,13)a_4=1265;$

$W(13,14)a_4 = 13a_4+55a_3+421a_2+15a_1+3785, M(13,14)a_4=1260;$

and finally $\quad M(225a_4, a_5) = 1472.$

Thus we have $M(3a_4, a_5) = 1472.$

7. Now consider any number $N > 3a_4$.

Let $N-3a_4 = A_r a_r + A_{r-1}a_{r-1} + A_{r-2}a_{r-2} + \ldots + A_5 a_5 + R,$
where $R < a_5$, $A_5 a_5 + R < a_6$, $A_6 a_6 + A_5 a_5 + R < a_7$, etc.

Then $N = A_r a_r + A_{r-1}a_{r-1} + A_{r-2}a_{r-2} + \ldots + A_5 a_5 + R',$
where $3a_4 < R' < a_5 + 3a_4$

Hence $$Min.(N) \leqslant 1475 + \sum_{i=5}^{r} A_i.$$

To find a suitable upper limit for A's, let

$$a_{i+1} - a_i < A_i a_i + A_{i-1}a_{i-1} + \ldots + A_5 a_5 + R < a_{i+1};$$

then $$A_i \leqslant [(a_{i+1} + a_{i-1})/a_i] - 1.$$

Moreover if

$$a_{i+j+1} - a_{i+j} < A_i a_i + A_{i-1}a_{i-1} + \ldots + A_5 a_5 + R, \; j = 0, 1, 2, \ldots, s;$$

then $$\sum_{j=1}^{s} A_{i+j} \leqslant 1.$$

If $a_{i+s+1}/a_{i+1} < 2$, then too $\sum_{j=1}^{s} A_{i+j} \leqslant 1.$

These observations can be used to prove that 1670 powers suffice for numbers up to 10^{48} if not beyond.

8. It would thus appear that the real Waring's Problem aims at finding two numbers G and H, such that

(i) every number $> G$ can be expressed as the sum of at the most H nth powers of primes; and

(ii) there exists an infinity of numbers $> G$, which require no less than H powers for their expression.

Vinogradow's[5] recent work shows that for every integer $n > 1$, it is possible to find a number $N_0(n)$ such that every integer $N \geqslant N_0$ can be expressed as a sum of $S(n)$ or less nth powers of primes. I am not aware if Vinogradow has given an explicit formula for $N_0(n)$. In any case, $N_0(n)$ is a sufficiently large integer. When a formula for $N_0(n)$ comes to be known, the methods of this paper, I believe, will be of use in taking

care of the integers $< N_0$. It will be sometime, however, before the conjecture of this paper can be proved in all its generality, and H may be quite hard to find.

REFERENCES

1. I. Chowla, *Proc. Indian Acad. Sci.*, **1** (1935), 451-453.
2. H. Gupta, *Proc. Indian Acad. Sci.*, **1** (1935), 789-794.

   ———————————, **4** (1936), 216-221.

   ———————————, **5** (1937), 381-384.

   ———, *Tohoku Math. Jour.*, **43** (1937), 11-16.
3. L. E. Dickson, *Bull. Amer. Math. Soc.*, **39** (1933), 701-727.
4. H. Gupta, *Amer. Jour. of Maths.*, **58** (1936), 573-576.
5. I. M. Vinogradow, *Comptes Rendus Acad. Sci., URSS*, **16** (1937) No. 3.

ON SOME RESULTS INVOLVING k_n-FUNCTIONS

By N. G. SHABDE, *College of Science, Nagpur.*

[Received 2 July 1938]

Introduction:—The object of this paper is to obtain a number of results involving Bateman's k_n-functions. We first note integrals of a general type, from which can be derived as particular cases several known results. Finally, we derive some k_n-function formulæ as particular cases of results obtained by Erdélyi in two recent papers*. We record these formulae, as they have not been known explicitly for the Bateman-functions.

1. A general integral involving $M_{k,m}(z)$-functions has been given by Erdélyi†. Since $M_{n,\frac{1}{2}}(t)=(-1)^{n-1}k_{2n}(t/2)$ we derive at once the general integral involving k_n-functions as

$$\int_0^\infty e^{-pz}z^q k_{2m_1}\left(\frac{\alpha_1 z}{2}\right)k_{2m_2}\left(\frac{\alpha_2 z}{2}\right)\ldots k_{2m_n}\left(\frac{\alpha_n z}{2}\right)dz$$

$$=(-1)^{m_1+m_2+\ldots+m_n-n}(\alpha_1\alpha_2\ldots\alpha_n)\left[p+\tfrac{1}{2}(\alpha_1+\ldots+\alpha_n)\right]^{-q-n-1}$$

$$\times\Gamma(q+n+1)F_A\Big(q+n+1;\,1-m_1,\ldots 1-m_n;\,2,2,\ldots,2;$$

$$\frac{\alpha_1}{p+\frac{1}{2}(\alpha_1+\ldots+\alpha_n)},\,\frac{\alpha_2}{p+\frac{1}{2}(\alpha_1+\ldots+\alpha_n)},\ldots,\frac{\alpha_n}{p+\frac{1}{2}(\alpha_1+\ldots+\alpha_n)}\Big) \tag{1.1}$$

* (i) "The Hankel transform of a product of Whittaker's functions", *Journ. Lond. Math. Soc.* 13 (1938), 146-154.

(ii) "On some expansions in Laguerre polynomials", *Ibid.*, 154-156.

† Über einige bestimmte integrale in denen die Whittakerschen $M_{k,m}$-functionen auftreten", *Math. Zeit.* 40 (1936), 693-702. There is a slight slip in this paper. The transformation quoted by Erdelyi on p. 702 should have the factor $(1-x)^{-2\beta}$ on the right-hand side, that is, it should be

$$F_2(\alpha,\beta,\beta,\alpha,\alpha,x,x)=\frac{1}{(1-x)^{2\beta}}F\left(\beta,\beta,\alpha,\frac{x^2}{(1-x)^2}\right).$$

Consequently, the value of the integral

$$\int_0^\infty e^{-\lambda x}[J_p(\lambda x)]^z dx$$

will also change correspondingly.

if $R(q+n+1)>0$, where F_A is the Lauricella hypergeometric function of n variables. This gives the operational representation

$$x^q k_{2m_1}\left(\frac{\alpha_1 x}{2}\right)\ldots k_{2m_n}\left(\frac{\alpha_n x}{2}\right)$$
$$\doteqdot (-)^{m_1+m_2+\ldots m_n-n} p(\alpha_1\ldots\alpha_n)\left[p+\tfrac{1}{2}(\alpha_1+\ldots+\alpha_n)\right]^{-q-n-1}$$
$$\times\Gamma(q+n+1)F_A\Bigg(q+n+1;\ 1-m_1,\ldots,\ 1-m_n;\ 2,\ 2,\ldots 2;$$
$$\frac{\alpha_1}{p+\frac{1}{2}(\alpha_1+\ldots+\alpha_n)},\ldots,\frac{\alpha_n}{p+\frac{1}{2}(\alpha_1+\ldots+\alpha_n)}\Bigg) \qquad (1.2)$$

Again we know that

$$\left.\begin{aligned} &x^{n/2}J_n(2\sqrt{x})\doteqdot p^{-n}e^{-\frac{1}{p}} \\ \text{or}\quad &e^{-s/p}p^{-n}\doteqdot x^{n/2}s^{-n/2}J_n(2\sqrt{xs})\end{aligned}\right\},\ n>-1. \qquad (1.3)$$

Write $\beta+p^{-1}$ for p in (1.1) throughout, multiply both sides by $p^{-\nu}$ and interpret the left-hand side regarding p as an operational parameter. This gives the result

$$x^{\nu/2}\int_0^\infty J_\nu(2\sqrt{xz})e^{-\beta z}z^{q-\nu/2}k_{2m_1}\left(\frac{\alpha_1 z}{2}\right)\ldots k_{2m_n}\left(\frac{\alpha_n z}{2}\right)dz$$
$$\doteqdot(-1)^{m_1+\ldots+m_n-n}(\alpha_1\ldots\alpha_n)p^{-\nu}\left[\beta+p^{-1}+\tfrac{1}{2}(\alpha_1+\ldots+\alpha_n)\right]^{-q-n-1}$$
$$\times\Gamma(q+1+n)F_A\Bigg\{q+n+1;\ 1-m_1,\ldots 1-m_n;\ 2,\ 2,\ldots 2;$$
$$\frac{\alpha_1}{\beta+p^{-1}+\frac{1}{2}(\alpha_1+\ldots+\alpha_n)},\ldots,\frac{\alpha_n}{\beta+p^{-1}+\frac{1}{2}(\alpha_1+\ldots\alpha_n)}\Bigg\},$$

where $\quad R(q+n+1)>0,\ p>0$ and $R(\nu)>-1. \qquad (1.4)$

2. Since $k_{2m}(x)\doteqdot\dfrac{2p(1-p)^{m-1}}{(1+p)^{m+1}}$, we have

$$\int_0^x k_{2n}(\xi)k_{2m}(x-\xi)d\xi$$
$$\doteqdot\frac{1}{p}\,\frac{2p(p-1)^{n-1}2p(p-1)^{m-1}(-1)^{m+n}}{(p+1)^{m+n+2}}$$
$$=\frac{4p(p-1)^{m+n-2}(-1)^{m+n}}{(p+1)^{m+n-\frac{1}{2}}}\left(\frac{1}{1+p}\right)^{5/2}$$
$$=\frac{(-1)^{m+n}4p(p-1)^{m+n-2}}{2^{5/2}(p+1)^{m+n-\frac{1}{2}}}\left(1-\frac{p-1}{p+1}\right)^{5/2}$$

$$= \frac{(-1)^{m+n}}{\sqrt{2}}\Bigg[\frac{(p-1)^{m+n-2}p}{(p+1)^{m+n-\frac{1}{2}}}$$

$$+\sum_{r=1}^{\infty}\frac{(-1)^r 5(5-2)\ldots(5-2r+2)}{2^r r!}\frac{p(p-1)^{m+n+r-2}}{(p+1)^{m+n+r-\frac{1}{2}}}\Bigg]$$

$$\doteqdot \frac{1}{\sqrt{2\pi}}\Bigg[\frac{2^{m+n-2}\Gamma(m+n-1)}{\Gamma(2m+2n-2)}D_{2m+2n-3}(2\sqrt{x})$$

$$+\sum_{r=1}^{\infty}\frac{(-1)^r 2^{m+n+r-2}\,\Gamma(m+n+r-1)D_{2m+2n+2r-3}(2\sqrt{x})}{\Gamma(2m+2n+2r-2)}\Bigg]$$

since $$\frac{p(p-1)^n}{(p+1)^{n+3/2}} \doteqdot \frac{(-1)^n 2^n\,\Gamma(n+1)}{\sqrt{\pi}\,\Gamma(2n+2)}D_{2n+1}(2\sqrt{x}). \qquad (2.1)$$

Hence we have

$$\int_0^x k_{2n}(\xi)k_{2m}(x-\xi)d\xi = \int_0^x k_{2m}(\xi)k_{2n}(x-\xi)d\xi$$

$$= \frac{1}{\sqrt{2\pi}}2^{m+n-2}\Bigg[\frac{\Gamma(m+n-1)}{\Gamma(2m+2n-2)}D_{2m+2n-3}(2\sqrt{x})$$

$$+\sum_{r=1}^{\infty}\frac{(-1)^r 2^r\,\Gamma(m+n+r-1)D_{2m+2n+2r-3}(2\sqrt{x})}{\Gamma(2m+2n+2r-2)}\Bigg] \qquad (2.2)$$

Again we know that

$$k_{2n}(x)e^{-x} \doteqdot -\frac{1}{\pi\sin n\pi}\sum_{m=0}^{\infty}\frac{p^{m+1}}{(m-n)(m+1-n)(2+p)^{m+1}} \qquad (2.3)$$

which can be derived from the known operational representation of $k_{2n}(x)$ for generalized n.

Also $$k_{2r}(x)e^{-x} \doteqdot \frac{2(-1)^{r-1}p^r}{(p+2)^{r+1}}. \qquad (2.4)$$

Hence

$$\int_0^x k_{2r}(\xi)k_{2n}(x-\xi)e^{-x}d\xi$$

$$\doteqdot \frac{2(-1)^r}{\pi\sin n\pi}\sum_{m=0}^{\infty}\frac{1}{(m-n)(m+1-n)}\frac{p^{r+m}}{(p+2)^{r+m+1}}\Bigg[1-\frac{p}{p+2}\Bigg]$$

$$= \frac{1}{2\pi\sin n\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(m-n)(m+1-n)}$$

$$\times\Bigg[\frac{2(-1)^{r+m-1}p^{r+m}}{(p+2)^{r+m+1}}+\frac{2(-1)^{r+m}p^{r+m+1}}{(p+2)^{r+m+2}}\Bigg]. \qquad (2.5)$$

This gives

$$\int_0^x k_{2r}(\xi)k_{2n}(x-\xi)d\xi = \int_0^x k_{2r}(x-\xi)k_{2n}(\xi)d\xi$$

$$= \frac{1}{2\pi \sin n\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m-n)(m+1-n)} \Big[k_{2m+2r}(x) + k_{2m+2r+2}(x) \Big] \tag{2.6}$$

3. As a particular case of (14) of my previous paper* we note that

$$\int_0^\infty J_0(2\sqrt{xs})k_{2n}(s)ds \doteqdot \frac{(-1)^{n-1}2p^2(1-p)^{n-1}}{(1+p)^{n+1}}$$

The right side is the operational representation of

$$(-1)^{n-1}\frac{d}{dx}k_{2n}(x).$$

So we get

$$\int_0^\infty J_0(2\sqrt{xs})k_{2n}(s)ds = (-1)^{n-1}\frac{d}{dx}k_{2n}(x). \tag{3.1}$$

Again from an integral for $k_{2n}(x)$, for $R(n) < 1$, we can see that

$$k_{2n}(p)e^p \doteqdot \frac{\sin n\pi}{n\pi}(1+2/x)^n. \tag{3.2}$$

Also
$$k_{2m}(x)e^{-x} \doteqdot \frac{2(-1)^{m-1}p^m}{(p+2)^{m+1}}. \tag{3.3}$$

Parseval's theorem, which states that if

$$f_1(x) \doteqdot \phi_1(p),\ f_2(x) \doteqdot \phi_2(p)$$

then
$$\int_0^\infty \frac{\phi_1 f_1}{x}dx = \int_0^\infty \frac{f_2\phi_1 dx}{x},$$

then gives

$$\frac{n\pi}{\sin n\pi}\int_0^\infty \frac{k_{2n}(x)k_{2m}(x)}{x}dx = (-1)^{m-1}2\int_0^\infty \frac{x^{m+n-1}}{(x+2)^{m+1-n}}dx$$

$$= \frac{(-1)^{m+1}\Gamma(m-n)}{\Gamma(m-n+1)}. \tag{3.3}$$

4. Here we record the formulæ derived from Erdélyi's papers as mentioned in the introduction.

* "On some k_n-function formulae", *Journ. Indian Math. Soc.* (New series), 2 (1937), 276-279.

$$\frac{\{k_{2n+2}(x/2)\}^2}{x}=\int_0^\infty \frac{J_2(2\sqrt{xy})\{k_{2n+2}(y/2)\}^2}{y}\,dy. \qquad (4.1)$$

$$\left.\begin{aligned}\frac{e^{x/2}}{x}k_{2n+2}(x/2) &= \frac{1}{2\pi i(n+1)}\int_{-\infty}^{(0+)} u^{-n-1}(1-u)^{1+n}e^{ux}du \\ &= \frac{(-1)^n}{2\pi i(n+1)}\int_{-1}^{(0+)} u^{-n-1}(1+u)^{1+n}e^{-ux}du.\end{aligned}\right\} \qquad (4.2)$$

$$\int_0^\infty J_{n+2}(2\sqrt{xy})e^{-y/2}y^{n/2}k_{2n+2}(y/2)\,dy$$
$$=(-1)^n e^{-x/2}x^{n/2}k_{2n+2}(x/2), \text{ if } n>-3. \qquad (4.3)$$

$$\int_0^\infty J_{2\mu+\nu+1}(2\sqrt{xy})\,y^{\mu+\frac{1}{2}\nu-\frac{1}{2}}e^{-\frac{1}{2}y}k_{2\nu+2}(y/2)\,dy$$
$$=\frac{\Gamma(2\mu+\nu+1)}{\Gamma(\nu+2)\,\Gamma(2\mu+1)}x^{\frac{1}{2}\nu}e^{-\frac{1}{2}x}M_{\mu+\nu+\frac{1}{2},\,\mu}(x)$$

if $R(2\mu+\nu+1)>1$ and $R(2\mu+\nu)>-1$. (4.4)

$$\int_0^\infty J_{2\mu+\nu+1}(2\sqrt{xy})\,y^{\mu+\frac{1}{2}\nu-\frac{1}{2}}e^{-\frac{1}{2}y}k_{2\nu+2}(y/2)\,dy$$
$$=\frac{(-1)^\nu}{\Gamma(\nu+2)}x^{\frac{1}{2}\nu}e^{-\frac{1}{2}x}W_{\mu+\nu+\frac{1}{2},\,\mu}(x)$$

if $R(2\mu+\nu+1)>1$ and $R(1-2\mu+\nu)>-\frac{1}{2}$. (4.5)

$$k_{2m}(z/2)k_{2n}(z/2)$$
$$=\frac{(-1)^{m+n-1}}{\Gamma(n+1)}\sum_{r=0}^{n}\frac{(1-m)_r(-1-n)_{n-r}}{r!\,(n-r)!\,(2)_r}z^r M_{m+n-r,\,r+\frac{1}{2}}(z). \qquad (4.6)$$

$$z^r M_{m+n-r,\,r+\frac{1}{2}}(z)$$
$$=(2)_r\sum_{s=0}^{2r}\frac{(-2r)_s}{s!}(-1)^{m+n-2r+s-1}k_{2m+2n-4r+2s}(z/2). \qquad (4.7)$$

$$k_{2m}(z/2)k_{2n}(z/2)$$
$$=\frac{1}{\Gamma(n+1)}\sum_{p=0}^{2n} l_p(-1)^{n-p}k_{2m+2n-2p}(z/2)$$

where $l_p=(-1)^{n+p}\sum\limits_{(\frac{1}{2}p\leqslant r\leqslant n)}\frac{(2r)!\,(1-m)_r(-1-n)_{n-r}}{p!\,r!\,(n-r)!\,(2r-p)!}$. (4.8)

$$(-1)^{m_1+m_2+\ldots+m_n} \frac{(m_1+1)(m_2+1)\ldots(m_n+1)}{l_1\ldots l_n z^n} e^{z(l_1+\ldots+l_n)/2}$$

$$\times k_{2m_1}\left(\frac{l_1 z}{2}\right) k_{2m_2}\left(\frac{l_2 z}{2}\right)\ldots k_{2m_n}\left(\frac{l_n z}{2}\right) = \sum_{s=0}^{m_1+\ldots+m_n} c_s L_s^{(\alpha)}(z)$$

where

$$c_s = \frac{(2)_{m_1}}{m_1!}\frac{(2)_{m_2}}{m_2!}\ldots\frac{(2)_{m_n}}{m_n!}$$

$$\times F_A(\alpha+1\,;\,-m_1,\ldots,-m_n,-s\,;\,2,2,\ldots 2,\alpha+1\,;\,l_1,l_2,\ldots l_n,1). \tag{4.9}$$

Taking $\alpha = 1$ in (4.9) we get

$$\frac{(-1)^{m_1+\ldots+m_n}}{l_1\ldots l_n z^{n-1}} e^{z(l_1+l_2+\ldots+l_n-1)/2}$$

$$\times k_{2m_1}\left(\frac{l_1 z}{2}\right)\ldots k_{2m_n}\left(\frac{l_n z}{2}\right) = \sum_{s=0}^{m_1+\ldots+m_n} c_s(-1)^s(s+1)k_{2s+2}(z/2). \tag{4.10}$$

INTEGRAL RELATIONS BETWEEN THE k-FUNCTION WITH NON-INTEGRAL INDEX AND CONFLUENT HYPERGEOMETRIC FUNCTIONS

By N. A. SHASTRI, *College of Science, Nagpur.*

[Received 21 July 1938]

In a recent paper[1] some properties of the k-function with non-integral index were established by the author. It is the object of this note to connect this function with Whittaker's function and its particular cases like Sonine's Polynomial, the Incomplete Gamma-function and the Pearson-Cunningham function. Incidentally, as will be seen, we get an integral equation for the k-function.

We have[2]

$$e^{\frac{1}{2}x} x^{k'-1} W_{k,m}(x) = \frac{(-)^{k-k'}\Gamma(k+m+\frac{1}{2})\,\Gamma(k-m+\frac{1}{2})}{2\pi i\,\Gamma(k+k')}$$

$$\times \int_C e^{-\frac{1}{2}s} s^{-k-1} (x+s)^{k+k'-1} W_{k',\,\pm m}(s)\,ds, \qquad (1)$$

provided

(a) $k' = n - \frac{1}{2} \pm m$ where n is a positive integer or zero,

(b) $\pm m - k - \frac{1}{2}$ is not an integer or zero

(c) C is the contour $(0+,\ \infty)$.

Throughout the discussion we will take the positive sign for m. Put $m=\frac{1}{2}$, $k'=\alpha+\frac{1}{2}$, and $k=n$ (not an integer) in (1). From (a) we see that α must be of the form $2r+1/2$. The equation (1) changes into

$$e^{\frac{1}{2}x} x^{\alpha-\frac{1}{2}} W_{n,\frac{1}{2}}(x) = \frac{(-)^{n-\alpha-\frac{1}{2}}\,\Gamma(n+1)\,\Gamma(n)}{2\pi i\,\Gamma(n+\alpha+\frac{1}{2})}$$

$$\times \int_C e^{-\frac{1}{2}s} s^{-n-1} (x+s)^{n+\alpha-\frac{1}{2}} W_{\alpha+\frac{1}{2},\frac{1}{2}}(s)\,ds. \qquad (2)$$

Now

$$k_{2n}\left(\frac{x}{2}\right)=-\frac{1}{2\pi i}\frac{e^{-\frac{1}{2}x}}{n}\int_{\infty}^{0+}e^{-t}(-t)^{-n}(x+t)^{n}dt$$

$$=\frac{1}{\Gamma(1+n)}W_{n,\frac{1}{2}}(x), \tag{3}$$

From (2) and (3) we get the following results immediately:

$$e^{\frac{1}{2}x}x^{\alpha-\frac{1}{2}}k_{2n}\left(\frac{x}{2}\right)=\frac{(-)^{n-\alpha-\frac{1}{2}}\Gamma(n)\,\Gamma(\alpha+\frac{3}{2})}{2\pi i\,\Gamma(n+\alpha+\frac{1}{2})}$$

$$\times\int_{C}e^{-\frac{1}{2}s}s^{-n-1}(x+s)^{n+\alpha-\frac{1}{2}}k_{2\alpha+1}(s/2)ds, \tag{4}$$

an integral equation satisfied by the k-function.

$$e^{\frac{1}{2}x}x^{\alpha-\frac{1}{2}}k_{2n}\left(\frac{x}{2}\right)=\frac{(-)^{n-\alpha-\frac{1}{2}}\Gamma(n)}{2\pi i\,\Gamma(n+\alpha+\frac{1}{2})}$$

$$\times\int_{C}e^{-\frac{1}{2}s}s^{-n-1}(x+s)^{n+\alpha-\frac{1}{2}}W_{\alpha+\frac{1}{2},\frac{1}{2}}(s)ds, \tag{5}$$

and

$$e^{\frac{1}{2}x}x^{\alpha-\frac{1}{2}}W_{n,\frac{1}{2}}(x)=\frac{(-)^{n-\alpha-\frac{1}{2}}\Gamma(n+1)\,\Gamma(n)\,\Gamma(\alpha+\frac{3}{2})}{2\pi i\,\Gamma(n+\alpha+\frac{1}{2})}$$

$$\int_{C}e^{-\frac{1}{2}s}s^{-n-1}(x+s)^{n+\alpha-\frac{1}{2}}k_{2\alpha+1}(s/2)ds. \tag{6}$$

In (4), (5) and (6), n is not an integer and α is of the form $2r+1/2$. We will now get the relation with the Pearson-Cunningham function. For this, we take the relation[3]

$$\omega_{\alpha,\beta}(x)=\frac{(-)^{\alpha-\frac{1}{2}\beta}}{\Gamma(\alpha-\frac{1}{2}\beta+1)}x^{-\frac{1}{2}(\beta+1)}e^{-\frac{1}{2}x}W_{\alpha+\frac{1}{2},\frac{1}{2}\beta}(x) \tag{7}$$

and put $\beta=1$ and use the transformed relation with (2) and (3). We have

$$e^{+\frac{1}{2}x}x^{\alpha-\frac{1}{2}}k_{2n}\left(\frac{x}{2}\right)=\frac{(-)^{n-2\alpha}\Gamma(n)\,\Gamma(\alpha+\frac{1}{2})}{2\pi i\,\Gamma(n+\alpha+\frac{1}{2})}$$

$$\times\int_{C}s^{-n}(x+s)^{n+\alpha-\frac{1}{2}}\omega_{\alpha,1}(s)ds. \tag{8}$$

Now put $m=\frac{1}{2}$, $k=\alpha+\frac{1}{2}$, and $k'=n$ in (1) where n is an integer and $2\alpha+1/2$ is not an integer or zero. We get by using with the resulting equation the relations (7) and (3)

$$e^x x^n \omega_{\alpha,1}(x) = \frac{(-)^{2\alpha-n}\Gamma(n+1)\,\Gamma(\alpha+\frac{3}{2})}{2\pi i\,\Gamma(\alpha+n+\frac{1}{2})}$$
$$\times\int_C e^{-\frac{1}{2}s} s^{-\alpha-\frac{1}{2}}(x+s)^{\alpha+n-\frac{1}{2}} k_{2n}(s/2)\,ds, \qquad (9)$$

where n is an integer and $2\alpha+1/2$ is not an integer or zero. Again if we take the relation[3]

$$T^{\nu}_{\beta}(x) = \frac{1}{\nu!\,(\beta+\nu)!} x^{-\frac{1}{2}(\beta+1)} e^{\frac{1}{2}x} W_{\nu+\frac{1}{2}\beta+\frac{1}{2},\,\frac{1}{2}\beta}(x)$$

where ν is an integer, and put $\beta = 1$ we get

$$T^{\nu}_{1}(x) = \frac{1}{\nu!\,(\nu+1)!} x^{-1} e^{\frac{1}{2}x} W_{\nu+1,\,\frac{1}{2}}(x).$$

Substitute in (1) $m = \frac{1}{2}$, $k' = \nu+1$ (ν an integer), and $k = n$ (n not an integer or zero). Using (3) and the above relation we will have

$$x^{\nu} e^{\frac{1}{2}x} k_{2n}\left(\frac{x}{2}\right) = \frac{(-)^{n-\nu-1}\Gamma(n)\,\Gamma(\nu+1)\,\Gamma(\nu+2)}{2\pi i\,\Gamma(n+\nu+1)}$$
$$\times\int_C e^{-s} s^{-n}(x+s)^{n+\nu} T^{\nu}_{1}(s)\,ds. \qquad (10)$$

This is the relation with Sonine's polynomial. Finally put $n = 1$ in the relation

$$\Gamma(n) - V(n, x) = x^{\frac{1}{2}(n-1)} e^{-\frac{1}{2}x} W_{\frac{1}{2}(n-1),\,\frac{1}{2}n}(x).$$

We get

$$1 - V(1, x) = e^{-\frac{1}{2}x} W_{0,\,\frac{1}{2}}(x).$$

Substitute in (1) $m = \frac{1}{2}$, $k' = 0$, and $k = n$ (not an integer or zero); we get with the help of (3) and the result obtained above

$$k_{2n}\left(\frac{x}{2}\right) = \frac{(-)^n e^{-\frac{1}{2}x} x}{2\pi i}\int_C [1 - V(1, s)]\, s^{-n-1}(x+s)^{n-1} ds, \qquad (11)$$

where n is not a positive integer or zero, a relation between the k-function and the Incomplete Gamma-function.

REFERENCES.

(1) SHASTRI—*Proc. Indian. Acad. Sci*, 1 (1935), 768-771.
(2) GIBB—*Proc. Edin. Math. Soc*, 34 (1916), 93-101.
(3) WHITTAKER & WATSON—*Modern Analysis*, (1927), 353.

RELATIONS BETWEEN SOME CONFLUENT HYPERGEOMETRIC FUNCTIONS

By N. A. SHASTRI, *College of Science, Nagpur.*

[Received 12 September 1938]

The object of this paper is to investigate certain properties of and relations between Weber's Parabolic Cylinder Functions and Bateman's k-function, which are both particular cases of Whittaker's $W_{k,m}$ function. Throughout this paper, methods of Operational Calculus are used.

1. A given function $h(x)$ is said to be the operational representation of another function $f(p)$ if

$$f(p) = p\int_0^\infty e^{-px}h(x)dx \tag{1.1}$$

provided the integral converges. The relation between the 'original' $h(x)$ and its 'image' $f(p)$ is denoted by

$$f(p) \doteqdot h(x).$$

We will require for our investigations the following theorems of the operational calculus[1]

$$f\left(\frac{p}{s}\right) \doteqdot h(sx) \text{ if } s = \text{const.} > 0 \tag{1.2}$$

$$\frac{p}{p+\alpha}f(p+\alpha) \doteqdot e^{-\alpha x}h(x) \tag{1.3}$$

$$p\left(-\frac{d}{dp}\right)^n\left[\frac{f(p)}{p}\right] \doteqdot x^n h(x),\ n > 0. \tag{1.4}$$

$$\int_0^\infty \frac{f(p)}{p}dp = \int_0^\infty \frac{h(x)}{x}dx; \tag{1.5}$$

the product theorem

$$\frac{1}{p}f_1(p)f_2(p) \doteqdot \int_0^x h_1(\xi)h_2(x-\xi)d\xi \tag{1.6}$$

where $h_1(x)$ and $h_2(x)$ are the originals of $f_1(p)$ and $f_2(p)$ respectively.

2. Connection between the k-function and Legendre function: We have[2]

$$k_{2n}(t) \doteqdot \frac{2p(1-p)^{n-1}}{(1+p)^{n+1}},\ n>0 \text{ if } t \doteqdot \frac{1}{p} \tag{2.1}$$

Hence by (1.2)—(1.5), we get

$$e^t k_{2n}(t) \doteqdot \frac{2(2-p)^{n-1}}{p^n}$$

$$e^{\frac{yt}{2}} k_{2n}\left(\frac{yt}{2}\right) \doteqdot \frac{y(y-p)^{n-1}}{p^n}$$

$$e^{-t+\frac{yt}{2}} k_{2n}\left(\frac{yt}{2}\right) \doteqdot \frac{py\,[y-p-1]^{n-1}}{[p+1]^{n+1}}$$

$$te^{-t+\frac{yt}{2}} k_{2n}\left(\frac{yt}{2}\right) \doteqdot -p\frac{d}{dp}\left[\frac{y[y-p-1]^{n-1}}{[p+1]^{n+1}}\right]. \tag{2.2}$$

Hence

$$\int_0^\infty e^{-t+\frac{yt}{2}} k_{2n}\left(\frac{yt}{2}\right)dt = -\int_0^\infty \frac{d}{dp}\left[\frac{y(y-p-1)^{n-1}}{(p+1)^{n+1}}\right]dp \tag{2.3}$$

$$= y(y-1)^{n-1}.$$

Put $y = x^2$ and differentiate both sides n times with respect to x. We obtain

$$\int_0^\infty e^{-t}\frac{d^n}{dx^n}\left[e^{\frac{x^2t}{2}} k_{2n}\left(\frac{x^2t}{2}\right)\right]dt$$

$$= \frac{d^n}{dx^n}(x^2-1)^n + \frac{d}{dx}\left[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\right]$$

$$= 2^n\Gamma(n+1)P_n(x) + 2^{n-1}\Gamma(n)P'_{n-1}(x), \tag{2.4}$$

using the relation

$$P_n(x) = \frac{1}{2^n\Gamma(n+1)}\frac{d^n}{dx^n}(x^2-1)^n.$$

Therefore

$$\frac{1}{2^{n-1}\Gamma(n)}\int_0^\infty e^{-t}\frac{d^n}{dx^n}\left\{e^{\frac{x^2t}{2}} k_{2n}\left(\frac{x^2t}{2}\right)\right\}dt = 2nP_n(x) + P'_{n-1}(x).$$

Equation (2.3) can be transformed into

$$\int_0^\infty \left[\frac{2}{y^2}e^{-\frac{2u}{y}}\right][e^u k_{2n}(u)]\,du = (y-1)^{n-1}.$$

Now in this equation put $y = x^2$ and differentiate both sides $(n-1)$ times with respect to x, we get

$$\int_0^\infty e^u k_{2n}(u) \frac{d^{n-1}}{dx^{n-1}}\left[\frac{2}{x^4} e^{-\frac{2u}{x^2}}\right] du = 2^{n-1}\Gamma(n) P_{n-1}(x). \quad (2.5)$$

The differentiations under the integral sign can be justified if x lies in the interval $0 < x < \sqrt{2}$.

3. We have from (2.2)

$$e^x k_{2n}(x) \doteqdot \frac{2(2-p)^{n-1}}{p^n} \text{ if } x \doteqdot \frac{1}{p}.$$

Therefore by using (1.2), we get

$$\begin{aligned} e^{xu} k_{2n}(xu) &\doteqdot \frac{2u}{p}\left[\frac{2u}{p} - 1\right]^{n-1} \\ &= \frac{2u(u-1)^{n-1}}{p}\left[1 + \frac{u}{u-1}\frac{2-p}{p}\right]^{n-1} \\ &= \sum_{r=0}^{n-1} \binom{n-1}{r} u^{r+1}(u-1)^{n-r-1}\frac{2(2-p)^r}{p^{r+1}}. \end{aligned}$$

Hence after the interpretation of the right hand side, we obtain

$$u^{-n} e^{xu} k_{2n}(xu) = \sum_{r=0}^{n-1} \binom{n-1}{r}\left(1 - \frac{1}{u}\right)^{n-r-1} k_{2r+2}(x) e^x.$$

This can be put in the form

$$\lambda^n e^{x(\frac{1}{\lambda}-1)} k_{2n}\left(\frac{x}{\lambda}\right) = \sum_{r=0}^{n-1} \binom{n-1}{r} (1-\lambda)^{n-r-1} k_{2r+2}(x). \quad (3.1)$$

This result holds for all positive values of λ. Results of analogous nature but under different conditions have been given by the author before[3].

From the operational formula

$$\frac{x^n}{\Gamma(1+n)} \doteqdot \frac{1}{p^n}, \; n \text{ a positive integer},$$

we have by (1.3)

$$\begin{aligned} \frac{e^{-x}x^n}{\Gamma(1+n)} &\doteqdot \frac{p}{(1+p)^{n+1}} \\ &= 2^{-n} \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{2p(1-p)^r}{(1+p)^{r+2}}. \end{aligned}$$

This gives after interpretation

$$\frac{2^n e^{-x} x^n}{\Gamma(1+n)} = \sum_{r=0}^{n-1} \binom{n-1}{r} k_{2r+2}(x)$$

$$= \sum_{r=1}^{n} \binom{n-1}{n-r} k_{2n-2r+2}(x). \qquad (3.2)$$

4. Substituting $\frac{t-1}{t+1} = -\frac{1}{p}$ in the generating function of Bateman's k-function

$$e^{y\frac{t-1}{t+1}} = \sum_0^\infty t^r k_{2r}(y),$$

we get

$$e^{-\frac{y}{p}} = \sum_0^\infty (-)^r \left(\frac{1-p}{1+p}\right)^r k_{2r}(y).$$

Therefore

$$\frac{2pe^{-\frac{y}{p}}}{(1+p)^2} = \frac{1}{p}\left[\frac{2pe^{-\frac{y}{p}}}{1+p} - \frac{2pe^{-\frac{y}{p}}}{(1+p)^2}\right]$$

$$= \sum_0^\infty (-)^r \frac{2p(1-p)^r}{(1+p)^{r+2}} k_{2r}(y). \qquad (4.1)$$

But

$$e^{-\frac{y}{p}} \doteqdot J_0(2\sqrt{xy})$$

$$\frac{p}{p+1} \doteqdot e^{-x},$$

Hence

$$\frac{p}{(1+p)^2} = p\left(-\frac{d}{dp}\right)\left(\frac{1}{p}\frac{p}{p+1}\right) \doteqdot xe^{-x}$$

Therefore using the product theorem and interpreting both sides of (4.1) with the help of the relations given above, we get

$$\sum_{r=0}^{\infty}(-)^r k_{2r}(y)k_{2r+2}(x)$$

$$=2\left[\int_0^x e^{-(x-\xi)}J_0(2\sqrt{y\xi})d\xi-\int_0^x (x-\xi)e^{-(x-\xi)}J_0(2\sqrt{y\xi})\,d\xi\right.$$

$$=2\int_0^x (1-x+\xi)e^{-(x-\xi)}J_0(2\sqrt{y\xi})d\xi$$

$$=2\int_0^x (1-t)e^{-t}J_0(2\sqrt{y(x-t)})dt. \tag{4.2}$$

5. Relations between the parabolic cylinder functions and the k-functions will now be obtained. Let us substitute the result

$$h(x)=e^{-\frac{x}{2}}k_{2n}\left(\frac{x}{2}\right)\doteqdot\frac{(-)^{n-1}p^n}{(1+p)^{n+1}}=f(p)$$

in the operational formula[4]

$$f(\sqrt{p})\doteqdot\frac{1}{\sqrt{\pi x}}\int_0^{\infty}e^{-\frac{s^2}{4x}}h(s)ds. \tag{5.1}$$

We get

$$\frac{1}{\sqrt{\pi x}}\int_0^{\infty}e^{-\frac{s^2}{4x}-\frac{s}{2}}k_{2n}\left(\frac{s}{2}\right)ds\doteqdot\frac{(-1)^{n-1}p^{\frac{n}{2}}}{(1+\sqrt{p})^{n+1}}$$

$$=\sum_{r=0}^{n-1}(-)^{n+r-1}\binom{n-1}{r}\frac{\sqrt{p}}{(1+\sqrt{p})^{r+2}}.$$

From this we have after interpreting with the help of the relation

$$\sqrt{\frac{\pi}{2}}\frac{\sqrt{p}}{(1+\sqrt{p})^m}\doteqdot(2x)^{\frac{1}{2}(m-1)}e^{\frac{1}{2}x}D_{-m}(\sqrt{2x})$$

the result

$$\int_0^{\infty}e^{-\frac{s^2}{4x}-\frac{s}{2}}k_{2n}\left(\frac{s}{2}\right)ds$$

$$=\sum_{r=0}^{n-1}(-)^{n+r-1}\binom{n-1}{r}(2x)^{\frac{r}{2}+1}e^{\frac{1}{2}x}D_{-r-2}(\sqrt{2x}). \tag{5.2}$$

Now

$$e^{\frac{x}{2}}k_{2n}\left(\frac{x}{2}\right)\doteqdot\frac{1}{p}\left(\frac{1}{p}-1\right)^{n-1} \tag{5.3}$$

Taking this relation twice, once with the index n and again with m, we get after applying the product theorem,

$$e^{\frac{x}{2}}\int_0^x k_{2m}\left(\frac{t}{2}\right)k_{2n}\left(\frac{x-t}{2}\right)dt \doteq p^{-3}\left(\frac{1}{p}-1\right)^{m+n-2} \tag{5.4}$$

$$\doteq\int_0^x (x-t)k_{2m+2n-2}\left(\frac{t}{2}\right)e^{\frac{t}{2}}dt. \tag{5.5}$$

The right side of (5.4) can be written as

(a) $\frac{1}{p}\,\frac{1}{p^{\frac{5}{2}}}\sqrt{p}\left[\frac{1}{p}-1\right]^{m+n-2}$

(b) $\frac{1}{p}\,\frac{1}{p^{\frac{3}{2}}}p^{-\frac{1}{2}}\left[\frac{1}{p}-1\right]^{m+n-2}$

and the resulting equations can be interpreted with the help of the operational relations

$$x^{-\frac{1}{2}}e^{\frac{1}{2}x}D_{2n}(\sqrt{2x})\doteq\frac{\sqrt{\pi}(2n)!}{2^n n!}\sqrt{p}\left(\frac{1}{p}-1\right)^n \tag{5.6}$$

$$e^{\frac{1}{2}x}D_{2n+1}(\sqrt{2x})\doteq\frac{\sqrt{\pi}(2n+1)!}{2^{n+\frac{1}{2}}n!}p^{-\frac{1}{2}}\left(\frac{1}{p}-1\right)^n \tag{5.7}$$

$$\frac{x^n}{\Gamma(1+n)}\doteq\frac{1}{p^n}$$

and the product theorem, giving us the results

$$e^{\frac{1}{2}x}\int_0^x k_{2m}\left(\frac{t}{2}\right)k_{2n}\left(\frac{x-t}{2}\right)dt$$

$$=\frac{2^{m+n-2}(m+n-2)!}{\sqrt{\pi}\,\Gamma(\frac{7}{2})(2m+2n-4)!}\int_0^x t^{-\frac{1}{2}}(x-t)^{\frac{5}{2}}e^{\frac{1}{2}t}D_{2m+2n-4}(\sqrt{2t})\,dt \tag{5.8}$$

$$=\frac{2^{m+n-\frac{3}{2}}}{\sqrt{\pi}}\,\frac{(m+n-2)!}{\Gamma(\frac{5}{2})(2m+2n-3)!}\int_0^x(x-t)^{\frac{3}{2}}e^{\frac{1}{2}t}D_{2m+2n-3}(\sqrt{2t})\,dt \tag{5.9}$$

$$5\int_0^x(x-t)^{\frac{3}{2}}e^{\frac{1}{2}t}D_{2m+2n-3}(\sqrt{2t})\,dt$$

$$=\sqrt{2}(2m+2n-3)\int_0^x t^{-\frac{1}{2}}(x-t)^{\frac{5}{2}}e^{\frac{1}{2}t}D_{2m+2n-4}(\sqrt{2t})\,dt. \tag{5.10}$$

(5.9) can be transformed by applying the duplication formula of the Gamma functions

$$2^{2n-1}\Gamma(n)\Gamma(n+\tfrac{1}{2}) = \sqrt{\pi}\,\Gamma(2n)$$

into

$$\int_0^x (x-t)^{\frac{3}{2}} e^{-\frac{1}{2}(x-t)} D_{2m+2n-3}(\sqrt{2t})\,dt$$

$$= 3\sqrt{\pi}\,2^{m+n-\frac{7}{2}}\,\Gamma(m+n-\tfrac{1}{2})\int_0^x k_{2m}\left(\frac{t}{2}\right)k_{2n}\left(\frac{x-t}{2}\right)dt. \quad (5.11)$$

If we take the combinations (5.3) (5.6) ; (5.3) (5.7) and proceed on similar lines, we get the relations

$$\int_0^x t^{-\frac{1}{2}} k_{2m}\left(\frac{x-t}{2}\right) D_{2n}(\sqrt{2t})\,dt$$

$$= \frac{2n\,!}{2^n n\,!}\int_0^x (x-t)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-t)} k_{2m+2n}\left(\frac{t}{2}\right)dt. \quad (5.12)$$

$$\int_0^x k_{2m}\left(\frac{t}{2}\right) D_{2n+1}(\sqrt{2(x-t)})\,dt$$

$$= \frac{(2n+1)\,!}{2^{n-\frac{1}{2}}n\,!}\int_0^x (x-t)^{\frac{1}{2}} e^{-\frac{1}{2}(x-t)} k_{2m+2n}\left(\frac{t}{2}\right)dt. \quad (5.13)$$

while the combination of (5.6) with (5.7) leads to

$$e^{\frac{x}{2}}\int_0^x t^{-\frac{1}{2}} D_{2n+1}(2\sqrt{x-t})\,D_{2m}(\sqrt{2t})\,dt$$

$$\doteq \frac{\pi}{2^{n+m+\frac{1}{2}}}\,\frac{2m\,!\,(2n+1)\,!}{n\,!\,m\,!}\,\frac{1}{p}\left(\frac{1}{p}-1\right)^{n+m}$$

$$= \frac{A}{p}\left(\frac{1}{p}-1\right)^{n+m} \quad \text{(say)}.$$

This gives rise to three results if the right hand side is taken as

$$\text{(a)}\ \frac{A}{p}\left(\frac{1}{p}-1\right)^{n+m} \qquad \text{(b)}\ \frac{A}{p}\frac{1}{\sqrt{p}}\sqrt{p}\left(\frac{1}{p}-1\right)^{n+m}$$

and

$$\text{(c)}\ \frac{A}{p}\sqrt{p}\,\frac{1}{\sqrt{p}}\left(\frac{1}{p}-1\right)^{n+m}$$

These results are

$$\int_0^x t^{-\frac{1}{2}} D_{2n+1}(\sqrt{2(x-t)})\,D_{2m}(\sqrt{2t})\,dt$$

$$= \frac{\pi}{2^{m+n+\frac{1}{2}}}\,\frac{(2m)\,!\,(2n+1)\,!}{m\,!\,n\,!}\,k_{2m+2n+2}\left(\frac{x}{2}\right) \quad (5.14)$$

$$= \frac{\sqrt{2}\,(2m)\,!\,(2n+1)\,!\,(m+n)\,!}{m\,!\,n\,!\,(2m+2n)\,!}$$

$$\times\int_0^x t^{-\frac{1}{2}}(x-t)^{\frac{1}{2}} e^{-\frac{1}{2}(x-t)} D_{2m+2n}(\sqrt{2t})\,dt \quad (5.15)$$

$$= \frac{(2m)!\,(2n+1)!\,(m+n)!}{m!\,n!\,(2m+2n+1)!}$$
$$\times\int_0^x (x-t)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-t)} D_{2m+2n+1}(\sqrt{2t})\,dt. \qquad (5.16)$$

From (5.15) and (5.16) we get

$$\int_0^x (x-t)^{-\frac{1}{2}} e^{\frac{1}{2}t} D_{2m+2n+1}(\sqrt{2t})\,dt$$
$$= \sqrt{2}(2m+2n+1)\int_0^x t^{-\frac{1}{2}}(x-t)^{\frac{1}{2}} e^{\frac{1}{2}t} D_{2m+2n}(\sqrt{2t})\,dt. \qquad (5.17)$$

6. If we take in the operational relation[5]

$$\frac{1}{p^{m-1}} f\left(\frac{1}{p}\right) \doteqdot \int_0^\infty \left(\frac{x}{s}\right)^{\frac{m}{2}} J_m(2\sqrt{xs})\,h(s)\,ds. \qquad (6.1)$$

$$f(p) = \frac{\sqrt{\pi}(2n+1)!}{2^n n!}\,\frac{p(1-p)^n}{(1+p)^{n+\frac{3}{2}}} \doteqdot D_{2n+1}(2\sqrt{x}) = h(x), \qquad (6.2)$$

we have

$$\frac{1}{p^{m-1}} f\left(\frac{1}{p}\right) = \frac{\sqrt{\pi}(2n+1)!}{2^n n!}(-)^n \frac{1}{p^{m-\frac{1}{2}}}\frac{1}{\sqrt{1+p}}\frac{p(1-p)^n}{(1+p)^{n+1}}$$
$$\doteqdot \int_0^\infty \left(\frac{x}{s}\right)^{\frac{m}{2}} J_m(2\sqrt{xs}) D_{2n+1}(2\sqrt{s})\,ds.$$

The case $m=1$ leads to the relation with the k-function, for

$$\int_0^\infty \left(\frac{x}{s}\right)^{\frac{1}{2}} J_1(2\sqrt{xs}) D_{2n+1}(2\sqrt{s})\,ds$$
$$\doteqdot (-)^n \frac{\sqrt{\pi}(2n+1)!}{2^n n!}\frac{p(1-p)^n}{(1+p)^{n+2}}\left(1+\frac{1}{p}\right)^{\frac{1}{2}}$$
$$= (-)^n \frac{\sqrt{\pi}(2n+1)!}{2^n n!}\frac{\sqrt{2}p(1-p)^n}{(1+p)^{n+2}}\left[1-\frac{1-p}{1+p}\right]^{-\frac{1}{2}}$$
$$= \frac{\sqrt{\pi}}{2^{n+\frac{1}{2}}}\frac{(2n+1)!}{n!}\sum_{r=0}^{\infty}(-)^{n+r}(-\tfrac{1}{2})_r \frac{2p(1-p)^{n+r}}{(1+p)^{n+r+2}}$$
$$\doteqdot \frac{\sqrt{\pi}}{2^{n+\frac{1}{2}}}\frac{(2n+1)!}{n!}\sum_{r=0}^{\infty}(-)^{n+r}(-\tfrac{1}{2})_r\, k_{2n+2r+2}(x)$$
$$= 2^{n+\frac{1}{2}}\Gamma(n+\tfrac{3}{2})\sum_{r=0}^{\infty}(-)^{n+r}(-\tfrac{1}{2})_r\, k_{2n+2r+2}(x),$$

using the duplication formula of the Gamma functions. Therefore

$$\int_0^\infty \left(\frac{x}{s}\right)^{\frac{1}{2}} J_1(2\sqrt{xs})D_{2n+1}(2\sqrt{s})ds$$

$$= 2^{n+\frac{1}{2}}\Gamma(n+\tfrac{3}{2}) \sum_{r=0}^{\infty} (-)^{n+r}(-\tfrac{1}{2})_r k_{2n+2r+2}(x). \qquad (6.3)$$

Similarly taking in (6.1) the relation

$$f(p) = \frac{\sqrt{\pi}(2n)!}{2^n n!} \frac{p(1-p)^n}{(1+p)^{n+\frac{1}{2}}} \doteqdot x^{-\frac{1}{2}} D_{2n}(2\sqrt{x}) = h(x). \qquad (6.4)$$

and proceeding as before, we get the result

$$\int_0^\infty \sqrt{x}\, s^{-1} J_1(2\sqrt{xs})D_{2n}(2\sqrt{s})ds$$

$$= \frac{\sqrt{\pi}(2n)!}{2^{n-\frac{1}{2}} n!} \sum_{r=0}^{\infty} (-)^{n+r}(-\tfrac{3}{2})_r k_{2n+2r+2}(x). \qquad (6.5)$$

REFERENCES

(1) Balth Van der Pol and K. F. Niessen: *Phil. Mag.* (7) 13 (1932), 539.

(2) Shastri: *Phil. Mag.* (7) 20 (1935), 472.

(3) Shastri: *loc. cit.*, 474; *Proc. Indian Acad. Sci.* 1 (1935), 770.

(4) P. Humbert: *Le Calcul Symbolique*, p. 28.

(5) Niessen: *Phil. Mag.* (7) 20 (1935), 980.

ON THE EXPANSION OF A FUNCTION IN A SERIES OF $q_n(z)$

By D. P. BANERJEE, *A. M. College, Mymensingh.*

[Received in revised form 9 August 1938]

1. The function $q_n(z)$ is used in certain problems connected with the spheroid, and is closely related to the Legendre function of the second kind. We have in fact

$$q_n(z)=i^{n+1}Q_n(iz)$$

$$=\int_0^\infty (z+\sqrt{z^2+1}\cosh\theta)^{-n-1}d\theta=\int_0^{(z^2+1)^{\frac{1}{2}}-z} h^n(1-2hz-h^2)^{-\frac{1}{2}}dh.* \tag{1}$$

In this paper is considered a method of expansion of a given function $f(z)$ in a series of $q_n(z)$, when $f(z)$ satisfies suitable conditions. A similar problem is briefly considered for the associated functions $q_n^m(z)$.

Let us assume

$$f(z)=\sum_0^\infty a_n q_n(z). \tag{2}$$

Let ϕ be a function such that

$$f(z)=\int_0^{(z^2+1)^{\frac{1}{2}}-z}\frac{\phi(h)dh}{(1-2hz-h^2)^{\frac{1}{2}}}. \tag{3}$$

Putting $\qquad h_1(1-h^2)=2h,$

we have $\qquad dh_1=2(1+h^2)dh/(1-h^2)^2.$

Putting $z_1=z^{-1}$, and writing

$$u(h_1)=\frac{(1-h^2)^2}{2(1+h^2)}\,\frac{\phi(h)h_1^{\frac{1}{2}}}{(2h)^{\frac{1}{2}}}, \tag{4}$$

we easily get from (3)

$$f\left(\frac{1}{z_1}\right)\Big/z_1^{\frac{1}{2}}=\int_0^{z_1}\frac{u(h_1)dh_1}{(z_1-h_1)^{\frac{1}{2}}}$$

* Whittaker and Watson, *Modern Analysis*, p. 313.

which is a particular case of Abel's integral equation.* Hence, if we assume that

$$\lim_{z_1\to 0} f\left(\frac{1}{z_1}\right)\Big/ z_1^{\frac{1}{2}} = 0 \quad i.e. \quad \lim_{z\to\infty} z^{\frac{1}{2}} f(z) = 0,$$

and that the differential coefficient of $f\left(\frac{1}{z_1}\right)\Big/ z_1^{\frac{1}{2}}$ is continuous, we get

$$u(h_1) = \frac{1}{\pi}\frac{d}{dh_1}\int_0^{h_1} \frac{f\left(\frac{1}{z_1}\right)dz_1}{z_1^{\frac{1}{2}}(h_1-z_1)^{\frac{1}{2}}}.$$

Equation (4) now gives the function $\phi(h)$. But, by Maclaurin's theorem,

$$\phi(h) = \sum_0^\infty \frac{\phi^{(n)}(0)}{n!} h^n.$$

Hence, using (3) and (1),

$$f(z) = \sum_0^\infty \frac{\phi^{(n)}(0)}{n!}\int_0^{(z^2+1)^{\frac{1}{2}}-z} \frac{h^n dh}{(1-2hz-h^2)^{\frac{1}{2}}}$$

$$= \sum_0^\infty \frac{\phi^{(n)}(0)}{n!} q_n(z).$$

2. As an interesting particular case, let us take $f(z) = z^{-m}$ where m is an integer greater than 1. The conditions specified above are satisfied. We get

$$\phi(h) = \frac{(1-h^2)^{\frac{1}{2}}}{\pi}\frac{d}{dh}\int_0^{2h/1-h^2} \frac{z_1^{m-\frac{1}{2}}dz_1}{\left(\frac{2h}{1-h^2}-z_1\right)^{\frac{1}{2}}}$$

$$= \frac{2(1-h^2)^{\frac{1}{2}}}{\pi}\frac{d}{dh}\left(\frac{2h}{1-h^2}\right)^m \int_0^{\frac{\pi}{2}} \sin^{2m}\theta \, d\theta$$

$$= \frac{2^m\Gamma(m+\frac{1}{2})}{(\pi)^{\frac{1}{2}}(m-1)!}\frac{h^{m-1}(1+h^2)}{(1-h^2)^{m+\frac{1}{2}}}$$

$$= \frac{2^m\Gamma(m+\frac{1}{2})}{\pi^{\frac{1}{2}}(m-1)!}\Big[h^{m-1} + (m+\tfrac{3}{2})h^{m+1} + \frac{(m+\frac{1}{2})\,(m+\frac{7}{2})}{2!}h^{m+3} + \dots\Big].$$

* Whittaker and Watson, *loc. cit.*, p. 223.

Hence,

$$z^{-m} = \frac{2^m\Gamma(m+\frac{1}{2})}{\pi^{\frac{1}{2}}(m-1)\,!}\Big[q_{m-1}(z) + (m+\tfrac{3}{2})q_{m+1}(z) + \frac{(m+\frac{1}{2})\,(m+\frac{7}{2})}{2\,!}q_{m+3}(z) + \ldots\Big].$$

This result which has been proved here when m is an integer greater than 1, is also true when $m = 1$, but the above method cannot be directly applied, since the derivative of $f\left(\frac{1}{z_1}\right)z_1^{-\frac{1}{2}}$ now becomes discontinuous when $z_1 = 0$. For this case, the method may be modified as follows:

Putting $f(z) = z^{-1}$, we have $f(z)/z = z^{-2}$. From the above,

$$z^{-2} = \frac{4\Gamma(2\frac{1}{2})}{\pi^{\frac{1}{2}}}\Big[q_1(z) + \tfrac{7}{2}q_3(z) + \frac{\frac{5}{2}\cdot\frac{11}{2}}{2\,!}q_5(z) + \ldots\Big].$$

$$z^{-1} = 3z\Big[q_1(z) + \tfrac{7}{2}q_3(z) + \frac{\frac{5}{2}\cdot\frac{11}{2}}{2\,!}q_5(z) + \ldots\Big].$$

Using the well-known recurrence formula

$$(2n-1)zq_{n-1}(z) = (n-1)q_{n-2}(z) - nq_n(z),$$
$$zq_0(z) = -q_1(z),$$

we get on simplification

$$z^{-1} = q_0(z) + \tfrac{5}{2}q_2(z) + \frac{3.9}{8}q_4(z) + \ldots\,.$$

This particular result is also deducible from a standard result due to Heine*, viz.

$$(t-z)^{-1} = \sum_{0}^{\infty}(2n+1)P_n(z)Q_n(t)$$

by putting $z = 0$, $t = iz$.

As another example, consider

$$f(z) = e^{a/z}/z^2$$

so that $f\left(\frac{1}{z_1}\right) = z_1^2 e^{az_1}$.

* Whittaker and Watson, *loc. cit.*, p. 316.

$$\phi(h)=\frac{(1-h^2)^{\frac{1}{2}}}{\pi}\frac{d}{dh}\left(\frac{h}{1-h}\right)^2 e^{\frac{ah}{1-h^2}}\int_0^\pi e^{-\frac{ah}{1-h^2}\cos\theta}(1-\cos\theta)^2 d\theta$$

$$=(1-h^2)^{\frac{1}{2}}\frac{d}{dh}\left(\frac{h}{1-h}\right)^2 e^{\frac{ah}{1-h^2}}\left[I_0\left(\frac{ah}{1-h^2}\right)+\frac{2I'}{a}\left(\frac{ah}{1-h^2}\right)+\frac{I''}{a^2}\left(\frac{ah}{1-h^2}\right)\right]^*$$

With this expression as $\phi(h)$,

$$e^{a/z}/z^2=\sum_0^\infty \frac{\phi^{(n)}(0)}{n!}q_n(z).$$

3. The associated function for $q_n(z)$ is

$$q_n^m(z)=(z^2+1)^{m/2}\frac{d^m}{dz^m}q_n(z)$$

$$=(z^2+1)^{m/2}\frac{d^m}{dz^m}\int_0^{(z^2+1)^{\frac{1}{2}}-z}\frac{h^n dh}{(1-2hz-h^2)^{\frac{1}{2}}}. \tag{5}$$

To express $F(z)$ in the form $\sum_0^\infty a_n q_n^m(z)$, we write

$$F(z)=(z^2+1)^{m/2}\frac{d^m}{dz^m}f(z)$$

$$=(z^2+1)^{m/2}\frac{d^m}{dz^m}\int_0^{(z^2+1)^{\frac{1}{2}}-z}\frac{\phi(h)dh}{(1-2hz-h^2)^{\frac{1}{2}}}.$$

We get, as in 1, under suitable conditions for $f(z)$

$$\phi(h)=\frac{2(1-h^2)^{\frac{1}{2}}}{\pi}\frac{d}{dh}\int_0^{\frac{2h}{1-h^2}}\frac{f\left(\frac{1}{z_1}\right)dz_1}{z_1^{\frac{1}{2}}\left(\frac{2h}{1-h^2}-z_1\right)^{\frac{1}{2}}}$$

For a given $F(z)$, we may take

$$f(z)=\int_0^z dz\int_0^z dz\ldots\int_0^z\frac{F(z)dz}{(z^2+1)^{m/2}}.$$

* Gray and Mathews: *Bessel Functions*, p. 46.

Having found $\phi(h)$, we use (5) and obtain the series in terms of $q_n^m(z)$.

As an example taking

$$F(z)=(z^2+1)^{m/2}\frac{d^m}{dz^m}\left(\frac{2}{z}\right),$$

we get

$$F(z)=q_0^m(z)+\tfrac{5}{2}q_2^m(z)+\frac{3.9}{2.4}q_4^m(z)+\ldots$$

My sincere thanks are due to the referee for helpful criticism.

INFINITE INTEGRALS INVOLVING WHITTAKER FUNCTIONS

By A. ERDÉLYI, *Brno, Czechoslovakia.*

[Received 19 December 1938]

1. Integrals of the type

$$\int_0^\infty x^{c-m-1} e^{-sx} M_{k,m}(px) W_{l,n}(qx)\, dx$$

are expressible in terms of hypergeometric functions of two variables of the type of Horn's series H_2*, if the real part of s is large enough to make the subject of integration vanish exponentially for $x \to \infty$. If s is situated on the frontier of the half plane of convergence, the expansion in hypergeometric series of two variables will no longer hold. The integral however exists, if certain additional conditions on the parameters are fulfilled. So we arrive at integrals of the type

$$J = \int_0^\infty x^{c-m-1} e^{-\frac{1}{2}(p-q)x} M_{k,m}(px) W_{l,n}(qx)\, dx.$$

Such integrals are also interesting from the point of view of the theory of hypergeometric series of *two* variables, for they represent the values of certain hypergeometric series of two variables in one of the singularities of these series.

It is the purpose of this paper to express J in terms of hypergeometric series of *one* variable, and to point out several special cases, which are of some importance.

2. To compute J we replace the $M_{k,m}$-function by the well-known integral representation†

$$e^{-\frac{1}{2}px} M_{k,m}(px) = \frac{\Gamma(2m+1)\,(px)^{m+\frac{1}{2}}}{\Gamma(\frac{1}{2}+k+m)\,\Gamma(\frac{1}{2}-k+m)} \int_0^1 u^{-\frac{1}{2}+k+m} (1-u)^{-\frac{1}{2}-k+m} e^{-pux}\, du,$$

$$\cdot [R(\tfrac{1}{2} \pm k+m) > 0,\ 2m+1 \neq 0, -1, -2, \ldots],$$

* J. Horn, 'Hypergeometrische Funktionen zweier Veranderlichen', *Math. Annalen* **105** (1931), 381-407.

† E. T. Whittaker and G. N. Watson, *Modern analysis*, 4th Edition, Cambridge (1927), p. 352, Example 1. With R ( ) we denote throughout the real part of the complex quantity in the brackets.

thus getting

$$J=\frac{\Gamma(2m+1)p^{m+\frac{1}{2}}}{\Gamma(\frac{1}{2}+k+m)\,\Gamma(\frac{1}{2}-k+m)}$$

$$\times\int_0^1\int_0^\infty u^{-\frac{1}{2}+k+m}(1-u)^{-\frac{1}{2}-k+m}x^{c-\frac{1}{2}}e^{-(pu-\frac{1}{2}q)x}W_{l,n}(qx)\,du\,dx. \tag{2.1}$$

It is allowed to invert the order of integration, supposing, that

$$|\arg p|<\tfrac{1}{2}\pi,\quad |\arg q|<\tfrac{3}{2}\pi,$$
$$R(\tfrac{1}{2}\pm k+m)>0,\quad R(c\pm n)>-1,\quad R(c+l)<-\tfrac{1}{2},$$
$$2m\neq-1,-2,-3,\ldots$$

Performing the integration with respect to x, we get, according to a result of Goldstein*

$$\int_0^\infty x^{c-\frac{1}{2}}e^{-(pu-\frac{1}{2}q)x}W_{l,n}(qx)\,dx$$
$$=\frac{\Gamma(c+n+1)\,\Gamma(c-n+1)}{\Gamma(c-l+\frac{3}{2})q^{c+\frac{1}{2}}}\,{}_2F_1\left[\begin{matrix}c+n+1,\ c-n+1\\ c-l+\frac{3}{2}\end{matrix};\ 1-\frac{pu}{q}\right],$$

which is, in virtue of the transformation formula

$${}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix};\ 1-z\right]=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\,{}_2F_1\left[\begin{matrix}a,\ b\\ a+b-c+1\end{matrix};\ z\right]$$
$$+\frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}z^{c-a-b}\,{}_2F_1\left[\begin{matrix}c-a,\ c-b\\ c-a-b+1\end{matrix};\ z\right] \tag{2.2}$$

for ordinary hypergeometric series, equal to

$$\frac{\Gamma(c+n+1)\,\Gamma(c-n+1)\,\Gamma(-c-l-\frac{1}{2})}{\Gamma(\frac{1}{2}-l+n)\,\Gamma(\frac{1}{2}-l-n)q^{c+\frac{1}{2}}}\,{}_2F_1\left[\begin{matrix}c+n+1,\ c-n+1\\ c+l+\frac{3}{2}\end{matrix};\ \frac{pu}{q}\right]$$
$$+\Gamma(c+l+\tfrac{1}{2})(pu)^{-c-l-\frac{1}{2}}q^{l}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}-l+n,\ \frac{1}{2}-l-n\\ \frac{1}{2}-c-l\end{matrix};\ \frac{pu}{q}\right].$$

Substituting this expression in (2.1), we see that under the assumption $|p/q|<1$, the hypergeometric series under the sign of integration are absolutely and uniformly convergent in the interval $0\leqslant u\leqslant 1$. Term by term integration is therefore permissible and yields

* S. Goldstein, 'Operational representation of Whittaker's confluent hypergeometric function and Weber's parabolic cylinder function', *Proc. London Math. Soc.* (2) 34 (1932), 103-25.

$$J=\int_0^\infty x^{c-m-1}e^{-\frac{1}{2}(p-q)x}M_{k,m}(px)W_{l,n}(qx)\,dx$$

$$=\frac{\Gamma(c+n+1)\,\Gamma(c-n+1)\,\Gamma(-\frac{1}{2}-c-l)}{\Gamma(\frac{1}{2}-l+n)\,\Gamma(\frac{1}{2}-l-n)}p^{m+\frac{1}{2}}q^{-c-\frac{1}{2}}$$

$$\times{}_3F_2\left[\begin{matrix}1+c+n,\ 1+c-n,\ \frac{1}{2}+k+m\\ \frac{3}{2}+c+l,\ 1+2m\end{matrix};\frac{p}{q}\right]$$

$$+\frac{\Gamma(\frac{1}{2}+c+l)\,\Gamma(-c+k-l+m)\,\Gamma(1+2m)}{\Gamma(\frac{1}{2}+k+m)\,\Gamma(\frac{1}{2}-c-l+2m)}p^{-c-l+m}q^{l}$$

$$\times{}_3F_2\left[\begin{matrix}\frac{1}{2}-l+n,\ \frac{1}{2}-l-n,\ -c+k-l+m\\ \frac{1}{2}-c-l,\ \frac{1}{2}-c-l+2m\end{matrix};\frac{p}{q}\right]. \tag{2.3}$$

This result, although proved only under some more restricting conditions, holds by the theory of analytic continuation, provided that

$$|\arg p|<\tfrac{1}{2}\pi,\quad |\arg q|<\tfrac{3}{2}\pi,\quad |p|<|q|;$$
$$R(c\pm n)>-1,\ R(c-m-k+l)<0,\ 2m\neq-1,-2,-3,\ldots \tag{2.4}$$

3. If we put $c=2m+n$, both the series ${}_3F_2$ reduce to ordinary hypergeometric series, thus yielding for the right of (2.3)

$$\frac{\Gamma(2m+2n+1)\,\Gamma(2m+1)\,\Gamma(-\frac{1}{2}-l-2m-n)}{\Gamma(\frac{1}{2}-l+n)\,\Gamma(\frac{1}{2}-l-n)}p^{m+\frac{1}{2}}q^{-2m-n-\frac{1}{2}}$$

$$\times{}_2F_1\left[\begin{matrix}\frac{1}{2}+k+m,\ 1+2m+2n\\ \frac{3}{2}+l+2m+n\end{matrix};\frac{p}{q}\right]$$

$$+\frac{\Gamma(\frac{1}{2}+l+2m+n)\,\Gamma(1+2m)\,\Gamma(k-l-m-n)}{\Gamma(\frac{1}{2}+k+m)\,\Gamma(\frac{1}{2}-l-n)}p^{-l-m-n}q^{l}$$

$$\times{}_2F_1\left[\begin{matrix}\frac{1}{2}-l+n,\ k-l-m-n\\ \frac{1}{2}-l-2m-n\end{matrix};\frac{p}{q}\right],$$

and this is, in virtue of the transformation formula (2.2), equal to

$$\frac{\Gamma(2m+1)\,\Gamma(2m+2n+1)\,\Gamma(k-l-m-n)}{\Gamma(\frac{1}{2}-l-n)\,\Gamma(1+k-l+m+n)}p^{m+\frac{1}{2}}q^{-2m-n-\frac{1}{2}}$$

$$\times{}_2F_1\left[\begin{matrix}\frac{1}{2}+k+m,\ 1+2m+2n\\ 1+k-l+m+n\end{matrix};1-\frac{p}{q}\right].$$

We therefore have the result

$$\int_0^\infty x^{m+n-1}e^{-\frac{1}{2}(p-q)x}M_{k,m}(px)W_{l,n}(qx)\,dx$$

$$=\frac{\Gamma(2m+1)\,\Gamma(2m+2n+1)\,\Gamma(k-l-m-n)}{\Gamma(\frac{1}{2}-l-n)\,\Gamma(1+k-l+m+n)}p^{m+\frac{1}{2}}q^{-2m-n-\frac{1}{2}}$$

$$\times{}_2F_1\left[\begin{matrix}\frac{1}{2}+k+m,\ 1+2m+2n\\ 1+k-l+m+n\end{matrix};1-\frac{p}{q}\right], \tag{3.1}$$

valid for

$$|\arg p| < \tfrac{1}{2}\pi, \quad |\arg q| < \tfrac{3}{2}\pi, \quad |p-q| < |q|;$$
$$R(m) > -\tfrac{1}{2}, \quad -\tfrac{1}{2} < R(m+n) < R(k-l).$$

Quite similarly we get, putting $c = 2m-n$ in (2.3),

$$\int_0^\infty x^{m-n-1} e^{-\frac{1}{2}(p-q)x} M_{k,m}(px) W_{l,n}(qx)\, dx$$
$$= \frac{\Gamma(2m+1)\,\Gamma(2m-2n+1)\,\Gamma(k-l-m+n)}{\Gamma(\frac{1}{2}-l+n)\,\Gamma(1+k-l+m-n)} p^{m+\frac{1}{2}} q^{-2m+n-\frac{1}{2}}$$
$$\times {}_2F_1\left[\begin{matrix}\frac{1}{2}+k+m,\ 1+2m-2n \\ 1+k-l+m-n\end{matrix};\ 1-\frac{p}{q}\right], \qquad (3.2)$$

valid for

$$|\arg p| < \tfrac{1}{2}\pi, \quad |\arg q| < \tfrac{3}{2}\pi, \quad |p-q| < |q|;$$
$$R(m) > -\tfrac{1}{2}, \quad -\tfrac{1}{2} < R(m-n) < R(k-l).$$

From the transformation theory of the hypergeometric function we can derive expressions for these integrals such as hold for other values of p, q. Using, for instance, the transformation formula

$${}_2F_1(a, b; c; z) = (1-z)^{-b} {}_2F_1\left(c-a, b; c; \frac{z}{z-1}\right)$$

on the right of (3.1), we readily get

$$\int_0^\infty x^{m+n-1} e^{-\frac{1}{2}(p-q)} M_{k,m}(px) W_{l,n}(qx)\, dx$$
$$= \frac{\Gamma(2m+1)\,\Gamma(2m+2n+1)\,\Gamma(k-l-m-n)}{\Gamma(\frac{1}{2}-l-n)\,\Gamma(1+k-l+m+n)} p^{-m-2n-\frac{1}{2}} q^{n+\frac{1}{2}}$$
$$\times {}_2F_1\left[\begin{matrix}\frac{1}{2}-l+n,\ 1+2m+2n \\ 1+k-l+m+n\end{matrix};\ 1-\frac{q}{p}\right], \qquad (3.3)$$

valid under similar conditions as (3.1), only the supposition $|p-q| < |q|$ being replaced by $|p-q| < |p|$. Applying the transformation (2.2) to the right of (3.3), we get an expression for the integral on the left of (3.3) valid for $|p| > |q|$. Thus, we can compute this integral for any values of $|p/q|$ in terms of ordinary hypergeometric series.

Putting $p = q = 1$ in (3.3), we at once get

$$\int_0^\infty x^{m+n-1} M_{k,m}(x)\, W_{l,n}(x)\, dx$$
$$= \frac{\Gamma(2m+1)\,\Gamma(2m+2n+1)\,\Gamma(k-l-m-n)}{\Gamma(\frac{1}{2}-l-n)\,\Gamma(1+k-l+m+n)} \qquad (3.4)$$

$[R(m) > -\tfrac{1}{2},\ -\tfrac{1}{2} < R(m+n) < R(k-l)]$.

We obtain a similar formula by putting $p=q=1$ in (3.2).

4. In this section some special cases of the integrals (3.1), (3.2) may be considered, in which at least one of the two Whittaker functions under the sign of integration reduces to modified Bessel functions, in virtue of the formulae

$$M_{o,m}(z) = \pi^{\frac{1}{2}} z^{\frac{1}{2}} \frac{\Gamma(2m+1)}{\Gamma(m+\frac{1}{2})} I_m(\tfrac{1}{2}z) \tag{4.1}$$

$$W_{o,n}(z) = \pi^{\frac{1}{2}} z^{\frac{1}{2}} K_n(\tfrac{1}{2}z). \tag{4.2}$$

We begin by putting $k=0$, $p=2$ in (3.1), thus obtaining, in consequence of (4.1),

$$\begin{aligned}\int_0^\infty & x^{m+n-\frac{1}{2}} e^{-(1-\frac{1}{2}q)x} I_m(x),\ W_{l,n}(qx)\,dx \\ &= \frac{\Gamma(m+\frac{1}{2})\,\Gamma(2m+2n+1)\,\Gamma(-l-m-n)}{\Gamma(\frac{1}{2}-l-n)\,\Gamma(m+n-l+1)\,\pi^{\frac{1}{2}}} 2^m q^{-2m-n-\frac{1}{2}} \\ &\quad \times {}_2F_1\left[\begin{matrix} m+\frac{1}{2},\ 2m+2n+1 \\ m+n-l+1 \end{matrix},\ 1-\frac{2}{q}\right]\end{aligned} \tag{4.3}$$

valid for

$$R(q) > 1;\ R(m) > -\tfrac{1}{2},\ -R(l) > R(m+n) > -\tfrac{1}{2}.$$

A similar formula is obtainable from (3.2).

The supposition $l=0$, $q=2$ in (3.1) yields, using (4.2),

$$\begin{aligned}\int_0^\infty & x^{m+n-\frac{1}{2}} e^{(1-\frac{1}{2}p)x} M_{k,m}(px)\, K_n(x)\,dx \\ &= \frac{\pi^{\frac{1}{2}}\,\Gamma(1+2m)\,\Gamma(1+2m+2n)\,\Gamma(k-m-n)}{\Gamma(\frac{1}{2}-n)\,\Gamma(1+k+m+n)\,2^{2m+n+1}} p^{m+1} \\ &\quad \times {}_2F_1\left[\begin{matrix} \frac{1}{2}+k+m,\ 1+2m+2n \\ 1+k+m+n \end{matrix},\ 1-\tfrac{1}{2}p\right]\end{aligned} \tag{4.4}$$

valid for

$$|1-\tfrac{1}{2}p| < 1;\ R(m) > -\tfrac{1}{2},\ R(k) > R(m+n) > -\tfrac{1}{2}.$$

An analogous formula is readily derived from (3.2).

As a last example of this kind, we put in (3.2) $k=l=0$ and replace p and q by $2a$ and $2b$ respectively. Thus we get, due to (4.1) and (4.2),

$$\int_0^\infty x^{m-n} e^{-(a-b)x} I_m(ax)\ K_n\ (bx)dx$$

$$=\frac{\Gamma(\frac{1}{2}+m)\ \Gamma(1+2m-2n)\ \Gamma(n-m)\ a^m}{\Gamma(1+m-n)\ \Gamma(\frac{1}{2}+n)\ 2^{m-n+1} b^{2m-n+1}}$$

$$\times {}_2F_1\left[\begin{matrix}\frac{1}{2}+m,\ 1+2m-2n \\ 1+m-n\end{matrix};\ 1-\frac{a}{b}\right]. \qquad (4.5)$$

valid for

$$|\arg a\ | <\tfrac{1}{2}\pi,\ |\arg b\ | <\tfrac{3}{2}\pi,\ |1-\frac{a}{b}\ | < 1;\ R\ (m)\ > -\tfrac{1}{2},$$

$$0 > R\ (m-n)\ > -\tfrac{1}{2}.$$

Some analogous formulæ, obtainable by the same assumptions on the values of the parameters in (3.3) and (3.4), are left to the reader.

5. Now we turn to integrals involving Weber's parabolic cylinder function $D_\nu(z)$, which is connected with Whittaker functions by means of the relations

$$\left.\begin{aligned} W_{\frac{1}{2}\nu+\frac{1}{4},\ \pm\frac{1}{4}}(\tfrac{1}{2}z^2) &= 2^{-\frac{1}{2}\nu-\frac{1}{4}} z^{\frac{1}{2}} D_\nu(z) \\ M_{\frac{1}{2}\nu+\frac{1}{4},\ \frac{1}{4}}(\tfrac{1}{2}z^2) &= \frac{\Gamma(-\frac{1}{2}\nu)}{\Gamma(-\frac{1}{2})} 2^{-\frac{1}{2}\nu-\frac{5}{4}} z^{\frac{1}{2}} [D_\nu(z)-D_\nu(-z)] \\ M_{\frac{1}{2}\nu+\frac{1}{4},\ -\frac{1}{4}} &= \frac{\Gamma(\frac{1}{2}-\frac{1}{2}\nu)}{\Gamma(\frac{1}{2})}\ 2^{\frac{1}{2}\nu-\frac{5}{4}} z^{\frac{1}{2}} [D_\nu(z)+D_\nu(-z)]. \end{aligned}\right\} \qquad (5.1)$$

Throughout in this section we replace x by $\frac{1}{2}y^2$.

At first we put $m=\frac{1}{4}$, $k=\frac{1}{2}\mu+\frac{1}{4}$ and $p=1$ in (3.1). Using (5.1), we obtain

$$\int_0^\infty y^{2n} e^{-\frac{1}{4}(1-q)y^2}\ [D_\mu(y)-D_\mu(-y)]\ W_{l,\ n}(\tfrac{1}{2}qy^2)dy$$

$$=\frac{\Gamma(1+\frac{1}{2}\mu)\ \Gamma(\frac{3}{2}+2n)\ \Gamma(\frac{1}{2}-\mu-l-n)}{\Gamma(\frac{3}{2}+\frac{1}{2}\mu-l+n)\ \Gamma(\frac{1}{2}-l-n)\ q^{n+1}}\ 2^{\frac{1}{2}\mu-n\frac{1}{2}}\ \sin\tfrac{1}{2}\mu\pi$$

$$\times {}_2F_1\left[\begin{matrix}1+\frac{1}{2}\mu,\ \frac{3}{2}+2n \\ \frac{3}{2}+\frac{1}{2}\mu-l+n\end{matrix};\ 1-\frac{1}{q}\right]; \qquad (5.2)$$

valid for

$$R(q)>\tfrac{1}{2};\ R(\tfrac{1}{2}\mu-l)\ > R(n) > -\tfrac{3}{4}.$$

Similarly the assumption $m=-\frac{1}{4}$, $k=\frac{1}{2}\mu+\frac{1}{4}$, $p=1$ in (3.1) yields

$$\int_0^\infty y^{2n-1} e^{-\frac{1}{4}(1-q)y^2} [D_\mu(y)+D_\mu(-y)]\, W_{l,\,n}(\tfrac{1}{2}qy^2)dy$$
$$= \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\mu)\,\Gamma(\frac{1}{2}+2n)\,\Gamma(\frac{1}{2}-l+\frac{1}{2}\mu-n)}{\Gamma(1-l+\frac{1}{2}\mu+n)\,\Gamma(\frac{1}{2}-l-n)q^n} 2^{\frac{1}{2}\mu+n}\cos\tfrac{1}{2}\mu\pi$$
$$\times {}_2F_1\left[\begin{matrix}\frac{1}{2}+\frac{1}{2}\mu,\ \frac{1}{2}+2n\\ 1-l+\frac{1}{2}\mu+n\end{matrix};1-\frac{1}{q}\right], \tag{5.3}$$

valid for

$$R(q) > \tfrac{1}{2};\ R(\tfrac{1}{2}\mu-l)+\tfrac{1}{2} > R(n) > -\tfrac{1}{4}.$$

We proceed by putting $n=\frac{1}{4}$, $l=-\frac{1}{2}\nu+\frac{1}{4}$, $q=1$ in (3.1). In consequence of (5.1) we get

$$\int_0^\infty y^{2m} e^{\frac{1}{4}(1-p)y^2} M_{k,\,m}(\tfrac{1}{2}py^2) D_{-\nu}(y)dy$$
$$= \frac{\pi^{\frac{1}{2}}\Gamma(2+4m)\,\Gamma(-\frac{1}{2}+k-m+\frac{1}{2}\nu)}{\Gamma(1+k+m+\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}\nu)\,2^{3m+\frac{1}{2}\nu+\frac{3}{2}}} p^{m+\frac{1}{2}}$$
$$\times {}_2F_1\left[\begin{matrix}\frac{1}{2}+k+m,\ \frac{3}{2}+2m\\ 1+k+m+\frac{1}{2}\nu\end{matrix};\ 1-p\right], \tag{5.4}$$

valid for

$$|1-p| < 0;\ \ R(m) > -\tfrac{1}{2},\ R(\tfrac{1}{2}-k+m-\tfrac{1}{2}\nu) < 0.$$

With the same values of l, n, q, (3.2) becomes

$$\int_0^\infty y^{2m-1} e^{\frac{1}{4}(1-p)y^2} M_{k,\,m}(\tfrac{1}{2}py^2) D_{-\nu}(y)dy$$
$$= \frac{\pi^{\frac{1}{2}}\Gamma(1+4m)\,\Gamma(k-m+\frac{1}{2}\nu)\ p^{m+\frac{1}{2}}}{\Gamma(\frac{1}{2}+k+m+\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}+\frac{1}{2}\nu)2^{3m+\frac{1}{2}\nu+1}}$$
$$\times {}_2F_1\left[\begin{matrix}\frac{1}{2}+k+m,\ \frac{1}{2}+2m\\ \frac{1}{2}+k+m+\frac{1}{2}\nu+1\end{matrix};\ 1-p\right], \tag{5.5}$$

valid for

$$|1-p| < 0;\ \ R(k+\tfrac{1}{2}\nu) > R(m) > -\tfrac{1}{4}.$$

Integrals involving products of parabolic cylinder functions occur, if we take for instance $m=\frac{1}{4}$, $k=\frac{1}{2}\mu+\frac{1}{4}$, $n=\frac{1}{4}$, $l=-\frac{1}{2}\nu+\frac{1}{4}$, $p=a^2$ and $q=b^2$ in (3.1). Using (5.1), we obtain in this case the formula

$$\int_0^\infty y e^{-\frac{1}{4}(a^2-b^2)y^2} [D_\mu(ay)-D_\mu(-ay)] D_{-\nu}(by)dy$$
$$= \frac{\Gamma(1+\frac{1}{2}\mu)2^{\frac{1}{2}\mu-\frac{1}{2}\nu+3} a\ \sin\frac{1}{2}\mu\pi}{(\mu+\nu+1)\,(\mu+\nu-1)b^3\,\Gamma(\frac{1}{2}\nu)}$$
$$\times {}_2F_1\left[\begin{matrix}1+\frac{1}{2}\mu,\ 2\\ \frac{3}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu\end{matrix};\ 1-\frac{a^2}{b^2}\right], \tag{5.6}$$

valid for

$$|\arg a| < \tfrac{1}{4}\pi,\ |\arg b| < \tfrac{3}{4}\pi,\ |a^2-b^2| < |b^2|\,;\ R(\mu+\nu) > 1.$$

When $m=-\frac{1}{4}$, $k=\frac{1}{2}\mu+\frac{1}{4}$, $n=\frac{1}{4}$, $l=-\frac{1}{2}\nu+\frac{1}{4}$, $p=a^2$ and $q=b^2$ are substituted in (3.1), they yield

$$\int_0^\infty e^{-\frac{1}{4}(a^2-b^2)y^2}[D_\mu(ay)+D_\mu(-ay)]D_{-\nu}(by)dy$$

$$=\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\mu)2^{\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{3}{2}}}{(\mu+\nu)b\,\Gamma(\frac{1}{2}\nu)}\cos\tfrac{1}{2}\mu\pi$$

$$\times{}_2F_1\left[\begin{matrix}\frac{1}{2}+\frac{1}{2}\mu,\ 1\\ 1+\frac{1}{2}\mu+\frac{1}{2}\nu\end{matrix};1-\frac{a^2}{b}\right], \tag{5.7}$$

valid for

$$|\arg a| < \tfrac{1}{4}\pi,\ |\arg b| < \tfrac{3}{4}\pi,\ |a^2-b^2| < |b^2|\,;\ R(\mu+\nu)>0.$$

We conclude this section putting $m=\frac{1}{4}$, $k=\frac{1}{2}\mu+\frac{1}{4}$, $n=\frac{1}{4}$, $l=-\frac{1}{2}\nu+\frac{1}{4}$, $p=a^2$ and $q=b^2$ in (3.2), thus obtaining, in virtue of (5.1),

$$\int_0^\infty e^{-\frac{1}{4}(a^2-b^2)y^2}[D_\mu(ay)-D_\mu(-ay)]D_{-\nu}(by)dy$$

$$=\frac{\Gamma(1+\frac{1}{2}\mu)2^{\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{3}{2}}a}{(\mu+\nu)b^2\,\Gamma(\frac{1}{2}+\frac{1}{2}\nu)}\sin\tfrac{1}{2}\mu\pi\ {}_2F_1\left[\begin{matrix}1+\frac{1}{2}\mu,\ 1\\ 1+\frac{1}{2}\mu+\frac{1}{2}\nu\end{matrix};1-\frac{a^2}{b^2}\right], \tag{5.8}$$

valid for

$$|\arg a| < \tfrac{1}{4}\pi,\ |\arg b| < \tfrac{3}{4}\pi,\ |a^2-b^2| < |b^2|\,;\ R(\mu+\nu)>0.$$

By addition and subtraction of (5.7) and (5.8), expressions for the integrals

$$\int_0^\infty e^{-\frac{1}{4}(a^2-b^2)y^2}D_\mu(\pm ay)D_{-\nu}(by)dy$$

in terms of hypergeometric series can be derived. Putting especially $a=b=1$, we get

$$\int_0^\infty D_\mu(\pm y)D_\nu(y)dy$$

$$=\frac{\pi.2^{\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}}}{\mu-\nu}\left[\frac{1}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu)\,\Gamma(-\frac{1}{2}\nu)}\mp\frac{1}{\Gamma(-\frac{1}{2}\mu)\,\Gamma(\frac{1}{2}-\frac{1}{2}\nu)}\right]. \tag{5.9}$$

* *Modern analysis*, § 14.53.
* *Modern analysis*, § 16.5.

6. In this section we shall discuss some integrals involving Laguerre polynomials. With r, s we denote in the following paragraphs non-negative integers. When $\pm k = \frac{1}{2}+m+r$ or $l = \frac{1}{2}+n+s$, the confluent hypergeometric series terminate, and Whittaker functions are expressible in terms of Laguerre polynomials, these being the only cases, in which Whittaker functions are expressible in "finite form".* We have

$$\left.\begin{aligned} M_{\pm(\frac{1}{2}+m+r),\,m}(z) &= \frac{r!\,\Gamma(2m+1)}{\Gamma(2m+r+1)} z^{m+\frac{1}{2}} e^{\mp\frac{1}{2}z} L_r^{(2m)}(\pm z), \\ W_{\frac{1}{2}+n+s,\,\pm n}(z) &= (-)^s s!\, z^{n+\frac{1}{2}} e^{-\frac{1}{2}z} L_s^{(2n)}(z). \end{aligned}\right\} \qquad (6.1)$$

At first, we put $m = \frac{1}{2}\alpha$, $k = \frac{1}{2}+\frac{1}{2}\alpha+r$, $q=1$ in (3.1). Thus we get, using (6.1),

$$\int_0^\infty x^{\alpha+n-\frac{1}{2}} e^{-(p-\frac{1}{2})x} L_r^{(\alpha)}(px) W_{l,n}(x)\,dx$$
$$= \frac{\Gamma(\alpha+r+1)\,\Gamma(\alpha+2n+1)\,\Gamma(\frac{1}{2}-l-n+r)}{r!\,\Gamma(\alpha-l+n+r+\frac{3}{2})\,\Gamma(\frac{1}{2}-l-n)}$$
$$\times {}_2F_1\left[\begin{matrix}\alpha+r+1,\ \alpha+2n+1 \\ \alpha-l+n+r+\frac{3}{2}\end{matrix};\ 1-p\right], \qquad (6.2)$$

valid for

$$|1-p|<1,\quad R(\alpha)>-1,\quad R(\alpha+2n)>-1.$$

When we put $p = 1$ in this formula, there occurs a special case of an integral discussed by R. S. Varma.†

We proceed, putting $m = \frac{1}{2}\alpha$, $k = -\frac{1}{2}-\frac{1}{2}\alpha-r$, $p = -1$, $q = 1/h$ in (3.1), and replacing x by hy. Thus, using (6.1), we obtain

$$\int_0^\infty y^{\alpha+n-\frac{1}{2}} e^{\frac{1}{2}y} L_r^{(\alpha)}(hy) W_{l,n}(y)\,dy$$
$$= \frac{\Gamma(\alpha+r+1)\,\Gamma(\alpha+2n+1)\,\Gamma(-\alpha-l-n-r-\frac{1}{2})}{r!\,\Gamma(\frac{1}{2}-l+n-r)\,\Gamma(\frac{1}{2}-l-n)}$$
$$\times {}_2F_1\left[\begin{matrix}-r,\ \alpha+2n+1 \\ \frac{1}{2}-l+n-r\end{matrix};\ 1+h\right], \qquad (6.3)$$

valid for

$$R(\alpha)>-1,\quad R(\alpha+2n)>-1,\quad R(\alpha+l+n+r)<-\tfrac{1}{2}.$$

Again, set $n = \frac{1}{2}\beta$, $l = \frac{1}{2}+\frac{1}{2}\beta+s$ and $q = 1$ in (3.1), and make use of (6.1), thus getting

* A. Erdélyi, 'Ueber die Integration der Whittakerschen Differential gleichung in geschlossener Form', *Monatsh. Math. Phys.* 46 (1937), 1-9.

† R. S. Varma, 'Some infinite integrals involving Weber's parabolic cylinder functions', *Journ. Indian Math. Soc.* New Series 3 (1938), 25-33, last equation on p. 26.

$$\int_0^\infty x^{\beta+m-\frac{1}{2}}e^{-\frac{1}{2}px}M_{k,m}(px)L_s^{(\beta)}(x)\,dx$$

$$=(-)^s\frac{\Gamma(2m+1)\,\Gamma(2m+\beta+1)\,\Gamma(k-m-\beta-s-\frac{1}{2})}{s!\,\Gamma(-\beta-s)\,\Gamma(\frac{1}{2}+k+m-s)}p^{m+\frac{1}{2}}$$

$$\times{}_2F_1\left[\begin{matrix}\frac{1}{2}+k+m,\ 2m+\beta+1\\ \frac{1}{2}+k+m-s\end{matrix};1-p\right], \qquad (6.4)$$

valid for

$$|1-p|<1;\quad R(\beta+2m)>-1,\quad R(\beta+m-k+s)<-\tfrac{1}{2}.$$

The last example of this kind we obtain by the assumptions $m=\frac{1}{2}\alpha$, $k=\frac{1}{2}+\frac{1}{2}\alpha+r$, $n=\frac{1}{2}\beta$, $l=\frac{1}{2}+\frac{1}{2}\beta+s$ and $q=1$ in (3.1). There results

$$\int_0^\infty x^{\alpha+\beta}e^{-px}L_r^{(\alpha)}(px)L_s^{(\beta)}(x)\,dx$$

$$=(-)^{r+s}\frac{\Gamma(\alpha+r+1)\,\Gamma(\beta+s+1)\,\Gamma(\alpha+\beta+1)}{r!\,s!\,\Gamma(\alpha+r-s+1)\,\Gamma(\beta-r+s+1)}$$

$$\times{}_2F_1\left[\begin{matrix}\alpha+r+1,\ \alpha+\beta+1\\ \alpha+r-s+1\end{matrix};1-p\right], \qquad (6.5)$$

valid for

$$|1-p|<1,\quad R(\alpha+\beta)>-1.$$

This integral was evaluated in a different manner by K. Mayr.*

7. We conclude by quoting a few more infinite integrals, which are special cases of those considered hitherto.

Putting $l=0$ in (5.2) and (5.3) respectively, the reader will, by means of (4.2), immediately check the integrals

$$\int_0^\infty y^{2n+1}e^{-\frac{1}{4}(1-q)y^2}[D_\mu(y)-D_\mu(-y)]K_n(\tfrac{1}{4}qy^2)\,dy$$

$$=\frac{\pi^{\frac{1}{2}}\Gamma(1+\frac{1}{2}\mu)\,\Gamma(\frac{3}{2}+2n)\,\Gamma(\frac{1}{2}\mu-n)}{\Gamma(\frac{3}{2}+\frac{1}{2}\mu+n)\,\Gamma(\frac{1}{2}-n)q^{n+\frac{3}{2}}}2^{\frac{1}{2}\mu+n+1}\sin\tfrac{1}{2}\mu\pi$$

$$\times{}_2F_1\left[\begin{matrix}1+\frac{1}{2}\mu,\ \frac{3}{2}+2n\\ \frac{3}{2}+\frac{1}{2}\mu+n\end{matrix};1-\frac{1}{q}\right], \qquad (7.1)$$

valid for

$$R(q)>\tfrac{1}{2};\quad \tfrac{1}{2}R(\mu)>R(n)>-\tfrac{3}{4},$$

and

* K. Mayr, 'Integraleigenschaften der Hermiteschen und Laguerreschen Polynome', *Math. Zeit.* 39 (1935), 597-604.

$$\int_0^\infty y^{2n} e^{-\frac{1}{4}(1-q)y^2} [D_\mu(y) + D_\mu(-y)] K_n(\tfrac{1}{4}qy^2)\, dy$$

$$= \frac{\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}+\frac{1}{2}\mu)\, \Gamma(\frac{1}{2}+2n)\, \Gamma(\frac{1}{2}+\frac{1}{2}\mu-n)}{\Gamma(1+\frac{1}{2}\mu+n)\, \Gamma(\frac{1}{2}-n) q^{n+\frac{1}{2}}} 2^{\frac{1}{2}\mu+n+\frac{1}{2}} \cos \tfrac{1}{2}\mu\pi$$

$$\times {}_2F_1\left[\begin{matrix} \frac{1}{2}+\frac{1}{2}\mu,\ \frac{1}{2}+2n \\ 1+\frac{1}{2}\mu+n \end{matrix};\ 1-\frac{1}{q}\right], \tag{7.2}$$

valid for

$$R(q) > \tfrac{1}{2};\quad \tfrac{1}{2}R(\mu)+\tfrac{1}{2} > R(n) > -\tfrac{1}{4}.$$

Similarly, putting $k=0$ in (5.4) and (5.5) respectively, and using (4.1), we obtain the infinite integrals

$$\int_0^\infty y^{2m+1} e^{\frac{1}{4}(1-p)y^2} I_m(\tfrac{1}{4}py^2) D_{-\nu}(y)\, dy$$

$$= \frac{\Gamma(\frac{1}{2}+m)\, \Gamma(\frac{3}{2}+2m)\, \Gamma(\frac{1}{2}\nu-m-\frac{1}{2})}{\Gamma(1+m+\frac{1}{2}\nu)\, \Gamma(\frac{1}{2}\nu)\, \pi^{\frac{1}{2}} 2^{\frac{1}{2}\nu-m}} p^m\, {}_2F_1\left[\begin{matrix} \frac{1}{2}+m,\ \frac{3}{2}+2m \\ 1+m+\frac{1}{2}\nu \end{matrix};\ 1-p\right], \tag{7.3}$$

valid for

$$|1-p|<1,\quad \tfrac{1}{2}R(\nu) > R(m)+\tfrac{1}{2} > 0,$$

and

$$\int_0^\infty y^{2m} e^{\frac{1}{4}(1-p)y^2} I_m(\tfrac{1}{4}py^2) D_{-\nu}(y)\, dy$$

$$= \frac{\Gamma(\frac{1}{2}+m)\, \Gamma(\frac{1}{2}+2m)\, \Gamma(\frac{1}{2}\nu-m)}{\Gamma(\frac{1}{2}+m+\frac{1}{2}\nu)\, \Gamma(\frac{1}{2}+\frac{1}{2}\nu)\pi^{\frac{1}{2}}} 2^{m-\frac{1}{2}\nu-\frac{1}{2}} p^m$$

$$\times {}_2F_1\left[\begin{matrix} \frac{1}{2}+m,\ \frac{1}{2}+2m \\ \frac{1}{2}+m+\frac{1}{2}\nu \end{matrix};\ 1-p\right], \tag{7.4}$$

valid for

$$|1-p| < 1,\quad \tfrac{1}{2}R(\nu) > R(m) > -\tfrac{1}{4}.$$

The two last-mentioned relations are, by virtue of the transformation (2.2) of the ordinary hypergeometric series, equivalent to certain results of R. S. Varma.*

Again, put $k=l=0$ in (6.2) and (6.4) respectively. This yields

* R. S. Varma, 'Extensions of some self-reciprocal functions', *Jour. Indian Math. Soc.* New Series **2** (1937), 269-75. Last equations on pp. 274 and 275.

$$\int_0^\infty y^{\alpha+n} e^{(h-\frac{1}{2})y} L_r^{(\alpha)}(hy) K_n(\tfrac{1}{2}y)\, dy$$

$$= \frac{\Gamma(\alpha+r+1)\,\Gamma(\alpha+2n+1)\,\Gamma(\frac{1}{2}-n+r)\pi^{\frac{1}{2}}}{r!\,\Gamma(\alpha+n+r+\frac{3}{2})\,\Gamma(\frac{1}{2}-n)} \times {}_2F_1\left[\begin{matrix}\alpha+r+1,\ \alpha+2n+1\\ \alpha+n+r+\frac{3}{2}\end{matrix};1-h\right], \tag{7.5}$$

valid for

$$|1-h|<1;\quad R(\alpha)>-1,\quad R(\alpha+2n)>-1,$$

and

$$\int_0^\infty y^{\beta+m} e^{-\frac{1}{2}px} I_m(\tfrac{1}{2}px) L_s^{(\beta)}(x)\, dx$$

$$= (-)^s \frac{\Gamma(\frac{1}{2}+m)\,\Gamma(1+\beta+2m)\,\Gamma(-\frac{1}{2}-\beta-m-s)}{s!\,\Gamma(-\beta-s)\,\Gamma(\frac{1}{2}+m-s)\pi^{\frac{1}{2}}} p^m \times {}_2F_1\left[\begin{matrix}\frac{1}{2}+m,\ 1+\beta+2m\\ \frac{1}{2}+m-s\end{matrix};1-p\right], \tag{7.6}$$

valid for

$$|1-p|<1;\quad R(m)+s+\tfrac{1}{2}<-R(\beta)<R(2m+1).$$

We proceed putting $m=\frac{1}{2}\alpha$, $k=\frac{1}{2}+\frac{1}{2}\alpha+r$ in (5.4) and (5.5) respectively, and using the first equation (6.1) with the upper signs. We arrive at

$$\int_0^\infty y^{2\alpha+1} e^{\frac{1}{4}(1-2p)y^2} L_r^{(\alpha)}(\tfrac{1}{2}py^2) D_{-\nu}(y)\, dy$$

$$= \frac{(\alpha+r+1)\,\Gamma(\alpha+\frac{3}{2})\,\Gamma(\frac{1}{2}\nu+r)}{r!\,\Gamma(\alpha+\frac{1}{2}\nu+r+\frac{3}{2})\,\Gamma(\frac{1}{2}\nu)} 2^{\alpha-\frac{1}{2}\nu}\, {}_2F_1\left[\begin{matrix}\alpha+r+1,\ \alpha+\frac{3}{2}\\ \alpha+\frac{1}{2}\nu+r+\frac{3}{2}\end{matrix};1-p\right], \tag{7.7}$$

valid for

$$|1-p|<1,\quad R(\alpha)>-1,$$

and

$$\int_0^\infty y^{2\alpha} e^{\frac{1}{4}(1-2p)y^2} L_r^{(\alpha)}(\tfrac{1}{2}py^2) D_{-\nu}(y)\, dy$$

$$= \frac{\Gamma(\alpha+r+1)\,\Gamma(\alpha+\frac{1}{2})\,\Gamma(\frac{1}{2}\nu+r+\frac{1}{2})}{r!\,\Gamma(\alpha+\frac{1}{2}\nu+r+1)\,\Gamma(\frac{1}{2}\nu+\frac{1}{2})2^{\frac{1}{2}\nu-\alpha+\frac{1}{2}}}\, {}_2F_1\left[\begin{matrix}\alpha+r+1,\ \alpha+\frac{1}{2}\\ \alpha+\frac{1}{2}\nu+r+1\end{matrix};1-p\right], \tag{7.8}$$

valid for

$$|1-p|<1,\quad R(\alpha)>-\tfrac{1}{2}.$$

Finally, we put $m=\frac{1}{2}\alpha$, $k=-\frac{1}{2}-\frac{1}{2}\alpha-r$, and replace p by $-p$ in (5.4) and (5.5) respectively, using the first equation (6.1) with the lower signs. We obtain

$$\int_0^\infty y^{2\alpha+1} e^{\frac{1}{4}y^2} L_r^{(\alpha)}(\tfrac{1}{2}py^2) D_{-\nu}(y)\,dy$$

$$= \frac{\Gamma(\alpha+r+1)\,\Gamma(\alpha+\frac{3}{2})\,\Gamma(\frac{1}{2}\nu-\alpha-r-1)}{r!\,\Gamma(\frac{1}{2}\nu-r+\frac{1}{2})\,\Gamma(\frac{1}{2}\nu)2^{\frac{1}{2}\nu-\alpha}}\,{}_2F_1\left[\begin{matrix}-r,\ \alpha+\frac{3}{2}\\ \frac{1}{2}\nu-r+\frac{1}{2}\end{matrix};\ 1+p\right], \tag{7.9}$$

valid for

$$\tfrac{1}{2}R(\nu)-r>R(\alpha)+1>0,$$

and

$$\int_0^\infty y^{2\alpha} e^{\frac{1}{4}y^2} L_r^{(\alpha)}(\tfrac{1}{2}py^2) D_{-\nu}(y)\,dy$$

$$= \frac{\Gamma(\alpha+r+1)\,\Gamma(\alpha+\frac{1}{2})\,\Gamma(\frac{1}{2}\nu-\alpha-r-\frac{1}{2})}{r!\,\Gamma(\frac{1}{2}\nu-r)\,\Gamma(\frac{1}{2}\nu\frac{1}{2})2^{\frac{1}{2}\nu-\alpha+\frac{1}{2}}}\,{}_2F_1\left[\begin{matrix}-r,\ \alpha+\frac{1}{2}\\ \frac{1}{2}\nu-r\end{matrix};\ 1+p\right], \tag{7.10}$$

valid for

$$\tfrac{1}{2}R(\nu)-r>R(\alpha)+\tfrac{1}{2}>0.$$

With these relations we conclude the series of examples for special cases of the rather general formulæ obtained in **3**. Some other special cases of these formulæ, containing besides the functions concerned with in this paper, Hermite polynomials, Bateman's k-function, the Error function and related functions, as well as some special cases of the more general formula (2.3), shall probably be published next time. I mention only in passing, that the series on the right of (6.4) and (7.6) can easily be transformed into terminating hypergeometric series. Similarly as in **3**, we can, applying the transformation theory of the hypergeometric series, compute the integrals discussed in this paper, for any values of p, q in terms of ordinary hypergeometric functions.

ON A FUNDAMENTAL THEOREM OF GEOMETRY

By F. W. LEVI, *Hardinge Professor, Calcutta University.*

[Received 28 November 1938]

If the projective axioms of the space are satisfied, then Desargues' theorem holds. If on the other hand in a plane the projective two-dimensional axioms are satisfied and Desargues' theorem holds, then the plane can be embedded into a space in which the projective axioms hold. The proof of the first proposition is easy and elementary; the second proposition has been proved by D. Hilbert by an ingenious arithmetical method which has the highest importance for the investigation of the fundamentals of Geometry. In this paper an alternative proof will be given which is elementary and geometrical; it applies a simple idea of descriptive geometry. Although descriptive geometry has occasionally been considered from an axiomatic point of view, its methods have—as far as I know—never been utilised for axiomatic purposes. As Descriptive Geometry is rather neglected in India, this paper is written in such a manner that the reader may follow it without references to that branch of mathematics.

1. In every quadrangle $ABCD$ the three diagonal points will be denoted by E_1, E_2, E_3.

MAIN PROPOSITION. *If the two quadrangles $A'B'C'D'$ and $A''B''C''D''$ are projective such that the lines $A'A''$, $B'B''$, $C'C''$, $D'D''$ and $E'_1E''_1$ pass through the same point Y, but $E'_1E''_1$ is different from the four other lines, then $E'_2E''_2$ and $E'_3E''_3$ also pass through Y.*

It will be proved:

I. *If the plane projective axioms and Desargues' theorem hold, then the main proposition holds.*

II. *If the plane projective axioms and the main proposition hold, then the plane can be embedded in a three-dimensional space in which the projective axioms hold.*

As it can be proved in an elementary manner that Desargues' theorem is a consequence of the spacial projective axioms, it follows that if the plane projective axioms hold, the following three propositions are equivalent:

(*a*) The main proposition.

(*b*) Desargues' theorem.

(*c*) The plane can be embedded in a three-dimensional space satisfying the projective axioms.

2. Before proving the Theorems I and II, the main proposition may be interpreted from the point of view of descriptive geometry. If we represent a spacial quadrangle $ABCD$ by orthogonal projection in a horizontal and a vertical plane, and carry the two representations into the same plane by rotation in the usual manner, the two points A' and A'' representing A will be situated on a parallel to the y-axis, and the same holds for the pairs of points $B'B''$, $C'C''$, $D'D''$. If $ABCD$ is a plane quadrangle, the diagonal points E'_1 and E''_1 are situated on a parallel to the y-axis; in this case $E'_1E''_1$ represents the diagonal-point E_1 of $ABCD$, but a plane quadrangle $ABCD$ has three diagonal-points E_1, E_2, E_3, represented by pairs of points $E'_1E''_1$, $E'_2E''_2$, $E'_3E''_3$ each pair being situated on a parallel to the y-axis. If on the other hand $E'_1E''_1$ is parallel to the y-axis, but if this line contains no vertex of the two quadrangles, then $ABCD$ is plane. From this consideration the main proposition follows if Y is the point at infinity of the y-axis, but this restriction can be removed by projective generalisation.

The method adopted in this paper is the following. Let the projective axioms and the main proposition hold in a plane; let a point Y be marked and let the pairs of points on each line passing through Y be defined as points of a certain space. Points satisfying the conditions enunciated in the main proposition will be considered to be coplanar. Thus by following a way converse to descriptive geometry, a space S will be built up where all projective axioms hold and in which the given plane is embedded.

3. To prove Theorem I, suppose that in a plane the plane projective axioms and Desargues' theorem hold. Let the quadrangles $A'B'C'D'$ and $A''B''C''D''$ satisfy the suppositions of the main proposition, let $E'_1 = A'D' \times B'C'$, $E'_2 = A'B' \times C'D'$ and $E'_1E''_2$ be defined correspondingly. From $A'A'' \neq E'_1E''_1$, it follows that $A'A'' \neq D'D''$. Therefore $A'D' = A''D''$ if and only if $A' = A'' = A$ and $D' = D'' = D$ and therefore $E'_1 = E''_1 = E_1$.

In this case the triangles $B'C'E'_2$ and $B''C''E''_2$ are in perspective as corresponding sides meet in E_1, D, A. Hence $B'B'', C'C'', E'_2E''_2$ are concurrent, i.e. $E'_2E''_2$ passes through Y. Without loss of generality, we therefore suppose that the point of intersection $A'D' \times A''D'' = P$ exists, and by symmetry that $B'C' \times B''C'' = Q$ exists.

Let $A'B' = A''B''$, then either this line passes through Y in which case the theorem is obvious; or $A' = A'' = A$, $B' = B'' = B$ in which case the triangles $AD'D''$ and $BC'C''$ are in perspective as corresponding sides intersect in E'_1, E''_1, Y. Hence AB, $C'D'$, $C''D''$ are concurrent, i.e. $E'_2 = E''_2$. Hence—using symmetry—we suppose without loss of generality, that $A'B' \times A''B'' = R$ and $C'D' \times C''D'' = S$ exist. As $C'D'E'_1$ and $C''D''E''_1$ are in perspective, the points S, P, Q are collinear. As $A'B'E'_1$ and $A''B''E''_1$ are in perspective, the points R, P, Q are collinear.

If $P \neq Q$, then RPS are collinear, hence $A'D'E'_2$ and $A''D''E''_2$ are in perspective. Hence $E'_2E''_2$ passes through Y.

If $P = Q$, then $P = Q = E'_1 = E''_1 = E_1$. We can suppose that $A' \neq A''$, $B' \neq B''$, $C' \neq C''$, $D' \neq D''$ as from $A' = A''$ it would follow that $D' = D''$ and this case has been considered above. Let $A'A'' \neq C'C''$, then $A'C' \times A''C'' = U$ exists and is different from E_1.

As $A'C'B'$ and $A''C''B''$ are in perspective, UE_1R are collinear.
As $A'D'C'$ and $A''D''C''$,, ,, UE_1S ,, ,, .

Hence $E_1 = P$, R, S are collinear, and as above it follows that $E'_2E''_2$ passes through Y. The corresponding proposition holds for $B'B'' \neq D'D''$. There remains the case that $A'A''C'C''$ are collinear, $B'B''D'D''$ are collinear and $P = Q = E'_1 = E''_1 = E_1$. Then $A'B'' \times D'C'' = Z$ exists and is different from Y.

As $C'D'C''$ and $B'A'B''$ are in perspective, E'_2, Z, Y are collinear
As $C''D''D'$ and $B''A''A'$,, ,, E''_2, Y, Z ,, ,,

Hence $E'_2E''_2$ passes through Y in every case. The same holds for $E'_3E''_3$. Hence Theorem I.

4. Let H be a system of points and lines satisfying the following conditions:

(*a*) Two different points of H are connected by one and only one line of H.

(*b*) Two different lines of H intersect in one and only one point of H.

(*c*) To every line there exist at least three points situated on it and a point not situated on it.

(*d*) The main proposition holds in H.

An arbitrary point Y of H will be distinguished, as also one line passing through Y which will be called the "line at infinity". The other lines of H together with those points of H which are not situated on the line at infinity form a system Π. The lines of H passing through Y will be called *ordinates*; thus every point A of Π lies on one and only one ordinate (the ordinate of A). The line at infinity is therefore also an ordinate. Lines of Π passing through the same point at infinity will be said to be *parallel*. The points (and lines) of Π will in future be called Π-points (and Π-lines).

By the help of Π a *space* S will be constructed consisting of *S-points*, *S-lines* and *S-planes* satisfying the following conditions:

(1) Two different S-points are connected by one and only one S-line.

(2) To every S-line there exists more than one point situated on it, and there exists an S-point not situated on it.

(3) Three S-points not situated on the same S-line define one and only one S-plane passing through them.

(4) If two S-points lie on an S-plane, all the points of the S-line connecting them lie on that S-plane (the S-line lies on the S-plane).

(5) To every S-plane α, there exists an S-point not situated on α.

(6) Let the S-point A be not situated on the S-line g, but A and g be situated on the S-plane α, then there exists one and only one S-line which lies on α, passes through A, but does not intersect g (the parallel of g through A).

(7) If two S-planes have a common S-point, then there exists another common point of them.

In every space S satisfying these conditions there exist pencils of parallel planes and bundles of parallel lines. By introducing *ideal* lines and *ideal* points corresponding to these pencils and bundles, and one *ideal* plane containing all those elements, we can

extend the space S to the *projective* space S^*. If we replace in (1)-(5) and (7) the notion of S-element by S^*-element, these conditions hold for S^*. Furthermore, (6) can be replaced by (6*): two different S^*-lines situated in the same S^*-plane are intersecting in one and only one S^*-point.

This extension of a general Euclidean space to a general projective space is well known, and will therefore not be repeated here in detail. If the system Π can be shown to be isomorphic to the system of S-points and S-lines of any S-plane α, the system H is isomorphic to the system of S^*-points and S^*-lines of the extended S^*-plane α^*, the elements at infinity of H corresponding to the ideal elements of α^*. By the method of *identification* we can therefore embed H as a general projective plane into the general projective space S^*. To complete the proof, we have therefore to construct the space S satisfying the conditions (1)-(7), and to show that Π is isomorphic to the system of S-points and S-lines situated in any particular S-plane.

5. An ordered pair of Π-points A' and A'' lying on the *same ordinate* will be called an S-point (A', A''). A' is called the *first*, A'' the *second projection* of (A', A''). The two projections may also be identical; if not, we get different S-points by interchanging A' and A''. An ordered pair of Π-lines g', g'' is said to be an *ordinary* S-line $[g', g'']$ if neither g' nor g'' is an ordinate, g' being the first, g'' being the second *projection* of g', g''. By intersecting this pair of Π-lines by an ordinate, we get the S-points (A', A'') situated on the S-line $[g', g'']$. Hence there exists one and only one ordinary S-line connecting two S-points (A', A'') and (B', B'') if the ordinates of these two S-points are different. To connect two S-points with the same ordinate, *special* S-lines will be introduced later on. The ordinary S-lines $[g', g'']$ and $[h', h'']$ have a common point if either $g' = h'$ and the Π-lines g'', h'' intersect, or $g'' = h''$ and the Π-lines g', h' intersect, or $g' \times h' = E'$, $g'' \times h'' = E''$ and E', E'' have the same ordinate. If g' is equal or parallel to h' and g'' is equal or parallel to h'', then $[g', g'']$ and $[h', h'']$ are said to be *parallel*. Let $g' \neq h'$, $g'' \neq h''$ and let the S-lines $[g', g'']$, $[h', h'']$ intersect in an S-point $(E'_1 E''_1)$, then we introduce an *ordinary plane* α in the following manner:

α contains all the points lying on $[g', g'']$ and on $[h', h'']$. Let E'_2 be a Π-point neither situated on g' nor on h'. As (c) holds in H, there exists a Π-line p' through E'_2 which is not an ordinate

and which does not pass through E'_1; this line* intersects g' in A' and h' in B'. The ordinates of A' and B' are different and intersect g'' (or h'') in A'' (or B''). The Π-line $A''B'' = p''$ intersects the ordinate of E'_2 in E''_2. Let q' be a Π-line different from p', but satisfying the same conditions; on repeating the last construction replacing p' by q', we get the Π-points C', D', C'', D'' and the Π-line q''. The quadrangles $A'B'C'D'$ and $A''B''C''D''$ satisfy the conditions of the main proposition, hence p'' and q'' intersect on the ordinate of E'_2 i.e. in E''_2.

The construction of E''_2 is therefore independent of the particular choice of p'. The pair of intersecting S-lines $[g', g'']$, $[h', h'']$ generates therefore an association by pairs of the Π-points such that to every E' there is associated a Π-point E'' on the same ordinate and this E'' is associated to no point other than E'. The S-points (E', E'') are considered as points of the ordinary S-plane α. From the construction of E''_2 for an arbitrary E'_2, it follows that every point of $[p', p'']$ lies on α. We shall now prove:

If two points (F', F''), (G', G'') of an ordinary line $[k', k'']$ are situated in α, then every point of $[k', k'']$ is situated in α.

PROOF: If the Π-line $F'G'$ does not pass through E'_1, then we can put F' for E'_2 and k' for p' and the proposition follows from the above remark. The same holds if $F''G''$ does not pass through E''_1 as the connection between the projections of the points and lines of α is symmetric for the projections. Let therefore k' pass through E'_1 and k'' through E''_1; we can suppose without loss of generality $F' \neq E'_1$; let (H', H'') be an S-point of α, and H'' lie on k''. If H' is not situated on k', then $F'H' = k'_1$ does not pass through E'_1. Hence $[k'_1, k'']$ lies on α in contradiction to the fact that E''_1 lies on k'', but E'_1 does not lie on k'_1. H' is therefore a Π-point of k' and every S-point of $[k', k'']$ lies on α.

Let $(A'_1A''_1)$, $(A'_2A''_2)$, $(A'_3A''_3)$ be S-points, and let $A'_1A'_2A'_3$ and $A''_1A''_2A''_3$ be triangles in the Π*-plane, then there exists one and only one ordinary S-plane containing those three S-points.*

PROOF: Without loss of generality we can suppose that the ordinate of $(A'_1A''_1)$ is different from the ordinates (which may be equal) of the two other S-points. $[A'_1A'_2, A''_1A''_2] = [g', g'']$

* One of the points A', B' may be a point at infinity, the ordinate of this point is the line at infinity; in any case the line p'' is a Π-line which is not an ordinate and which therefore intersects the ordinate of E'_2 in a Π-point.

and $[A'_1A'_3, A''_1A''_3] = [h', h'']$ are therefore two intersecting ordinary S-lines; therefore there exists an ordinary plane generated by these two S-lines and containing those three S-points. On the other hand every ordinary S-plane containing the given triplet of S-points must contain $[g', g'']$ and $[h', h'']$ and all the lines connecting S-points of these S-lines.

Let now β be any ordinary S-plane containing the given S-points, and (E', E'') be an S-point of β. If E' is given, we find E'' by the help of a construction which depends only on $[g', g'']$ and $[h', h'']$. Hence $\beta = \alpha$.

Let $[a', a'']$ and $[b', b'']$ be two ordinary S-lines on the S-plane α. If a' and b' are intersecting, then a'' and b'' intersect in a Π-point of the same ordinate, and conversely; if therefore $b' || b''$, then is $a' || a''$.

The connection between the two projections of ordinary S-lines on an ordinary S-plane is therefore such that parallel pencils correspond to parallel pencils. Two ordinary S-lines on the same ordinary S-plane are either intersecting or parallel.

6. The S-points, ordinary S-lines and ordinary S-planes satisfy the seven conditions of **4** with certain restrictions only. To fill up the remaining gaps, *special* S-lines and *special* S-planes will be introduced. These special elements correspond to the "elements in special situation" of descriptive geometry.

The S-points which are situated on the same ordinary S-plane and have the same ordinate form *an S-line of the second kind.*

From this definition it follows that if (A', A'') and (B', B'') are two different points of an S-line of the second kind, then both inequalities $A' \neq B'$, $A'' \neq B''$ hold.

If $A' \neq B'$, $A'' \neq B''$, and if (A', A''), (B', B'') have the same ordinate y, then there exists one and only one S-line of the second kind connecting (A', A'') and (B', B'').

PROOF: Let (D', D'') be an S-point with an ordinate different from y, then there exists an ordinary S-plane α containing (A', A''), (B', B''), (D', D'') and therefore containing an S-line of the second kind which connects (A', A'') with (B', B''). If two ordinary S-planes γ_1 and γ_2 both containing (A', A'') and (B', B'') have a common point with an ordinate $\neq y$, they are identical. Let γ_1 and γ_2 be different. In both planes, (A', A'') and (B', B'') are connected by an S-line of the second kind, say Γ_1 and Γ_2 respectively.

Let (C', C_1) be an S-point of Γ_1 and (C', C_2) be an S-point of Γ_2, $C_1 \neq C_2$, let g' be a Π-line which passes through C' and is not an ordinate. Then there exists an ordinary S-line $[g', g_1]$ in γ_1 and an ordinary S-line $[g', g_2]$ in γ_2. g_1 passes through C_1, and g_2 through C_2. If g_1 and g_2 intersect in, say G'', then the ordinate of G'' is $\neq y$ and intersects g' in a Π-point G'. Then (G', G'') must be a common point of γ_1 and γ_2, but that is impossible. Hence g_1 and g_2 are parallel. Let h' be the parallel of g' through A' and h'' be the parallel of g_1 and g_2 through A'', then $[h', h'']$ is common to γ_1 and γ_2, which is impossible. Hence $C_1 = C_2$; this holds for every Π-point C' on y. The two S-lines of the second kind Γ_1 and Γ_2 are therefore equal. Hence the theorem. In § **1** the following lemma will be applied.

"Given a point at infinity X and a Π-line a' neither passing through X nor through Y; then there exists a (1, 1)-correspondence between the S-lines Γ of the second kind with a given ordinate y and the Π-lines a'' passing through neither X nor Y such that the points (A', A'') of $[a', a'']$ are projected from X into the points (B', B'') of the corresponding S-line Γ."

Proof: Let Γ be given by $(B'_1, B''_1)(B'_2, B''_2)$; put $XB'_1 \times a' = A'_1$, $XB'_2 \times a' = A'_2$, $A'_1 Y \times B''_1 X = A''_1$, $A'_2 Y \times B''_2 X = A''_2$, $A''_1 A''_2 = a''$. The ordinary S-plane containing (B'_1, B''_1) and $[B'_2 X, B''_2 X]$ contains Γ, $[a', a'']$ and the S-lines $[B'_i X, B''_i X]$ where (B'_i, B''_i) are points of Γ. Hence the S-points of Γ are the X-projections of the S-points of $[a', a'']$; to every Γ there corresponds therefore a Π-line a''. If a'' is given, we find the corresponding S-line Γ on the ordinate y by inverting the order of the steps of the proof. Hence the lemma.

The S-points with the first projection A' form an S-line (of the third kind)

$$[A', -].$$

The S-points with the second projection A'' form an S-line (of the fourth kind)

$$[-, A''].$$

The S-points with the ordinate y form an S-plane (of the second kind)

$$\{y\}$$

The S-points of which the first projection is a Π-line g' which not an ordinate, form an S-plane (of the third kind)

$$\{g', -\}$$

The S-points of which the second projection is a Π-line g'' which is not an ordinate form an S-plane (of the fourth kind)

$$\{-, g''\}$$

7. The S-points, S-lines, S-planes as defined in **5** and **6** will now be proved to form a space satisfying the seven conditions of 4.

(1) If (A', A'') and (B', B'') have different ordinates, then they are connected by an ordinary S-line; if they have the same ordinate, but $A' \neq B'$ and $A'' \neq B''$, they are connected by an S-line of the second kind; if $A' = B'$ they are connected by an S-line of the third kind; if $A'' = B''$, they are connected by an S-line of the fourth kind. In no case more than one connecting S-line exists.

(2) and (5) hold obviously.

(3) Let (A', A''), (B', B''), (C', C'') be three different S-points. If $A'B'C'$ and $A''B''C''$ form two triangles, then there exists one and only one ordinary S-plane connecting them (see **5**), but no S-plane of a different kind passes through those three S-points. If A', B', C' (or A'', B'', C'') lie on g' (or g''), but $A''B''C''$ (or $A'B'C'$) forms a triangle, then only the S-plane $\{g', -\}$ (or the S-plane $\{-, g''\}$) connects the three points. If the three S-points have the same ordinate y, but are not situated on the same S-line (of second, third, or fourth kind), then only the S-plane $\{y\}$ connects them.

(4) If (A', A'') and (B', B'') have different ordinates, $A'B' = h'$; $A''B'' = h''$, then the S-planes containing (A', A'') and (B', B'') are: the ordinary S-planes containing $[h', h'']$ (see **5**), the S-plane $\{h', -\}$ and the S-plane $\{-, h''\}$; all these planes contain $[h', h'']$. If the ordinate of both the two points is y, but $A' \neq B'$, $A'' \neq B''$, then every ordinary S-plane passing through those two S-points contains the connecting S-line of the second kind (see **6**); the same holds for the S-plane $\{y\}$. There is no S-plane of the third or fourth kind containing both the S-points. If $A' = B'$, then the two S-points are connected by $[A', -]$, and this S-line is contained in every S-plane containing the two S-points; i.e. in $\{y\}$ and in the S-planes $\{g', -\}$ where g' passes through A'. The corresponding statement holds for $A'' = B''$.

(6) There exists a (1, 1)-correspondence between the elements of an ordinary S-plane and their first projections, where the ordinate of an S-line of the second kind is considered as its projection. The first projection of the S-line connecting two S-points is the Π-line connecting the first projections of the two S-points, and if two S-lines intersect in an S-point, its first projection is the point of intersection of the first projections of the two S-lines. The (1, 1)-correspondence is therefore an *isomorphism.* (The same holds for the second projection). As (6) holds for Π, it holds also for the ordinary S-planes isomorphic to Π. Any S-plane $\{g', —\}$ is isomorphic to its second projection, and an S-plane $\{—, g''\}$ is isomorphic to its first projection. Hence (6) holds also in these S-planes. An S-plane $\{y\}$ contains only lines of the second, third and fourth kinds. Lines of different kinds in $\{y\}$ are always *intersecting*; two lines of the third (of the fourth) kind are non-intersecting. Through every point of $\{y\}$ there passes one and only one line of the third (or the fourth) kind. We have therefore to prove that to every particular S-line of the second kind in $\{y\}$, there exists in $\{y\}$ one and only one non-intersecting S-line of the second kind which passes through any S-point of $\{y\}$. For the proof we apply the lemma of **6.** By the lemma these S-lines are (1, 1)-represented by the lines a'', i.e. the lines of Π which are not passing through two particular points at infinity. Two S-lines of the second kind intersect if the corresponding lines a'' intersect. As the proposition holds for the system of lines a'', it holds also for the S-lines of the second kind in $\{y\}$. Hence (6) holds in every case.

(7) To show that if two S-planes intersect, they have more than one common point, we investigate the intersection of the different kinds of S-planes. $\{y\}$ intersects every ordinary S-plane in an S-line of the second kind and every S-plane of the third (the fourth) kind in an S-line of the third (the fourth) kind; $\{y\}$ does not intersect a different S-plane of the same kind.

$\{g', —\}$ and $\{h', —\}$ intersects in $[A', —]$ if $g' \times h' = A'$; they do not intersect if $g' || h'$. The corresponding statement holds for $[—, g'']$ and $[—, h'']$. $\{g', —\}$ and $\{—, g''\}$ intersect in

$[g', g'']$. $\{g', —\}$ intersects an ordinary S-plane in an ordinary S-line $[g, g'']$; the corresponding statement holds for S-planes $\{\dot{—}, g''\}$.

Let now α and β be ordinary S-planes, and (A', A'') be a common point of them. Let g' be a Π-line neither passing through Y, nor through A'. Then there exists in α an ordinary S-line $[g', g_1]$ and in β an ordinary S-line $[g', g_2]$. If $g_1 \times g_2 = G''$, then there exists an S-point $(G', G'')(A', A'')$ which is a common S-point of α and β. If $g_1 || g_2$, let $h' || g'$ pass through A', and let $h'' || g_1 || g_2$ pass through A''; then is $[h', h'']$ the line of intersection of α and β.

8. In **7** it has been shown that the space S satisfies the conditions (1)-(7) of 4. It can therefore (see **4**) be extended to a general projective space S^*. As it has been shown in **7** (6), every ordinary S-plane is isomorphic to Π, thus Π can be embedded in S, and H can be embedded in S^*. Hence Theorem II holds. Hence the three propositions (a), (b), (c) of **1** are equivalent.

REMARK. In the proof of Theorem II, only one particular centre of projection Y has been utilised. The main proposition may therefore be restricted to the case where Y is a particular point. It is often possible to narrow an axiom in a similar manner without loss of generality. *e.g.* in ordinary geometry the Parallel axiom can be replaced by the postulate that one triangle exists in which the sum of the angles is equal to* 2π. In our case we get the following corollary:

COROLLARY: *In every pencil of lines of a Non-Desarguesian plane, there exists a pair of perspective quadrangles not satisfying the main proposition.*

* For another example, see: Ruth Moufang, *Math. Ann.* Vol. 107, p. 124.

PLANE CURVES WITH PSEUDO-RHAMPHOID CUSPS

By CLIFFORD BELL, *University of California.*

[Received 10 September 1938]

A necessary condition that the parametric equations

$$x = f(T),\ y = g(T) \tag{1}$$

represent the plane algebraic curve $U(x, y) = 0$ is that $U[f(T), g(T)] \equiv 0$ in the parameter T. This condition merely states that the points given by the equations (1) are points of the curve $U(x, y) = 0$, and does not imply that to every point of the curve there corresponds a value of T.

The equations (1) will be said to give a *proper* parametric representation of the curve $U(x, y) = 0$, if the following conditions are satisfied. First, as T takes on all the real values from $-\infty$ to $+\infty$, all the points of $U(x, y) = 0$ are obtained, with the exception of a finite number of isolated points. Second, to each ordinary point of the curve there corresponds one and only one value of the parameter, and to each isolated point there corresponds a pair of conjugate complex values.

If any part or all of the above conditions are not satisfied, the equations (1) will be said to give an *improper* parametric representation of the curve $U(x, y) = 0$.

The equations which give an improper parametric representation of a plane curve, taken by themselves, represent a curve which may possess properties that are of more interest than the original curve. It is the purpose of this paper to study such curves and to show that the results of a recent paper* apply when proper definitions for the singular points are given.

In the following work it is assumed that equations (1) give a proper parametric representation of the curve $U(x, y) = 0$, and

* Clifford Bell, 'On a Determinant Function Involving the Parameter of a Plane Curve', *American Mathematical Monthly*, 64 (1937), 218-221. Hereafter this paper will be referred to as the *Monthly* paper.

hence, hereafter a reference to the curve (1) means the curve $U(x, y)=0$.

Consider the equations,

$$x=f[r(t)],\ y=g[r(t)], \tag{2}$$

obtained by substituting $r(t)$ for T in the equations (1). The nature of this representation will be determined according as the conditions for proper representation are or are not satisfied, when t instead of T is used as parameter.

To the curve (1) is associated the function, $H_{ij}(T)$, which was defined in the *Monthly* paper as being equal to the determinant $|f_i(T)\ g_j(T)|$, where the subscripts on f and g indicate the order of the derivatives taken with respect to T. For the curve (2) the corresponding function is obtained by taking the derivatives with respect to t.

The first, second, third and fourth derivatives* of x with respect to t from equations (2) are respectively f_1r_1, $f_1r_2+f_2r_1^2$, $f_1r_3+3f_2r_1r_2+f_3r_1^3$, $f_1r_4+f_2(4r_1r_3+3r_2^2)+6f_3r_1^2r_2+r_1^4f_4$, where the subscripts on r represent the order of the derivatives taken with respect to t. The corresponding derivatives of y with respect to t are obtained by substituting g for f in the above expressions. Hence it is obvious that $H_{1j}(t)$ has a factor $r_1(t)$. This means that if t_1 is a root of $r_1(t)=0$, $H_{1j}(t_1)\equiv 0$ in j. But, according to the *Monthly* paper, this means a multiple point of the cuspidal type at the point (t_1) if $H_{12}(t)=0$ in t. That this is not an actual multiple point on the curve (1) is immediately evident when one considers the fact that $r(t)$ may be chosen in such a way that its derivative vanishes at any preassigned real value of t. Hence every point on (1) would be a multiple point of the cuspidal type which is obviously impossible†. Nevertheless a point (t_1), at which $r_1(t)$ vanishes, must have certain properties characteristic of the parametric representation (2), even though it is an ordinary point on the curve (1).

Consider the parametric representation given by equations (2) when the equation $r_1(t)=0$ has at least one root, $t=t_1$. Suppose, further, that $r_2(t_1)\neq 0$. Hence the function $T=r(t)$ has a turning point at $t=t_1$, which means that in the neighbourhood

* Continuity and the existence of derivatives to the orders indicated are assumed.

† See Ganguli, *The Theory of Plane Curves*, University of Calcutta Press 1925, Vol. 1, p. 62.

of (t_1) at least two values of t give the same value of T. Thus to a point on the curve (1) corresponds at least two values of t, and hence (2) is an improper representation of the curve (1). The curve represented by equations (2), considered as a distinct curve, may now be described as the parameter t passes through the value t_1. Since $t = t_1$ gives a turning point of the function $T = r(t)$, the point (t) moves along the curve (1) until it reaches the point (t_1) where it stops, reverses its direction and returns along the same path on which it had arrived.

It is seen that (t_1) is a point similar in nature to an actual multiple point of the cuspidal type. The only difference is that for an actual cusp the return path is different. A point such as (t_1) will be called a *pseudo-cusp of the second kind* or a *pseudo-rhamphoid cusp*, provided $H_{12}(t) \not\equiv 0$. It should be noted that the two coincident branches of the curve are necessarily on the same side of the tangent line to the curve at (t_1). If $H_{12}(t) \equiv 0$, the equations (2) represent, according to the *Monthly* paper, a straight line. But due to the nature of this representation, the equations may represent only a part of the line. Now at (t_1) for this case, the two branches of the curve and the tangent line are coincident. Merely to distinguish this point from the other kind, it will be called a *pseudo-cusp of the first kind* and not because it has any characteristics peculiar to the actual cusp of the first kind. In fact as defined, the pseudo-cusp of the first kind may be considered as the limiting case of an actual cusp of either the first or the second kind.

The tangent line at the pseudo-rhamphoid cusp, for the curve under consideration, obviously has 4-point contact, considering of course that the curve consists of two coincident branches. By the *Monthly* paper, the tangent line at a point (t_1) has $(k+1)$-point contact if $H_{ij}(t_1) = 0$, $(i, j = 1, 2, \ldots, k)$ and $H_{ik+1}(t_1) \neq 0$, when i takes on at least one of the values $1, 2, \ldots, k$. Does this theorem hold for the particular curve (2) under consideration?

Computation yields the following results. $H_{ij}(t_1) = 0$, $(i, j = 1, 2, 3)$ and $H_{24}(t_1) = 3r_2^3(t_1)H_{12}(T_1)$, where T_1 is the value of T when $t = t_1$. Supposing that (T_1) is an ordinary point on the curve (1), $H_{12}(T_1) \neq 0$, and as $r_2(t_1) \neq 0$, $H_{24}(t_1) \neq 0$. This means 4-point contact by the theorem, and hence the answer to the above question is in the affirmative. If (T_1) is not an ordinary point on the curve (1), the order of contact of the tangent line should increase depending upon the type of the special point. Thus if (T_1) is an inflection point, the tangent line has 6-point contact at

(t_1) for the curve (2), and it is found that $H_{46}(t_1) \neq 0$, while $H_{ij}(t_1) = 0$, $(i, j = 1, 2, 3, 4, 5)$.

In the definite case under consideration, it was assumed that $r_2(t_1) \neq 0$, which assured a simple turning point at $t = t_1$ for the function $T = r(t)$. Now it is well known that if $r_s(t_1) = 0$ and $r_{s+1}(t_1) \neq 0$, the function $T = r(t)$ has a turning point at $t = t_1$ when s is odd, and an inflection point with a horizontal tangent when s is even. However, in either case $H_{1j}(t_1) \equiv 0$ in j, but only for the case of the turning point does a pseudo-cusp exist. The vanishing of $H_{1j}(t_1)$ should have been expected for the other case, for such an inflexion point may be considered as the limiting position of a maximum, a minimum and an ordinary inflection point as they approach coincidence. Thus such a point on the curve (2) is the limiting position of two adjacent pseudo-cusps, one pointing in one direction along the curve (1) and the other in the opposite direction. The order of contact of the tangent line at (t_1) may, in all cases, be determined by the quantities $H_{ij}(t_1)$.

To illustrate the above theory two examples will be given. First consider the curve,

$$x = t^4 - 8t^3 + 18t^2 - 8t + 1, \quad y = t^2 - 4t + 6. \tag{3}$$

The quantities $H_{ij}(t)$ are as follows: $H_{12}(t) = -16(t-2)^3$, $H_{13}(t) = -48(t-2)^2$, $H_{14}(t) = H_{23}(t) = -48(t-2)$, $H_{24}(t) = -48$, and for all other values of i and j, $H_{ij}(t)$ vanishes. From these quantities it is seen that the curve (3) is not a line and that 2 is the only value of t for which all the quantities $H_{1j}(t) \equiv 0$ in j. Likewise it is noticed that $H_{ij}(2) = 0$, $(i, j = 1, 2, 3)$ and $H_{24}(2) \neq 0$. Therefore $t = 2$ gives either an actual rhamphoid cusp or a pseudo-rhamphoid cusp.

It will be noted that to each ordinary point on the curve there correspond two values of the parameter. Thus, to the point (1, 6) correspond the parameters $t = 0$ and $t = 4$. This means that the equations (3) give an improper representation of some other curve. Hence x and y may be expressed as functions of $r(t)$ whose derivative is of the form $a(t-2)$. Therefore, by integration, $r(t)$ is of the form $a(t^2 - 4t)/2 + b$, where a and b are arbitrary. Choosing $a = 2$ and $b = 6$, and setting $r(t) = T$, the equations (3) become

$$x = (T-5)^2, \quad y = T. \tag{4}$$

In non-parametric form, this curve takes the form $x = (y-5)^2$, a parabola with vertex at (0, 5), axis $y = 5$ and with its opening

to the right. It is this curve for which equations (3) are an improper parametric representation.

Now, since $r_2(2) > 0$, the point (9, 2) for which $t = 2$, is a pseudo-rhamphoid cusp of the curve (3), and furthermore, as $t = 2$ gives a minimum value of $r(t)$, the smallest value of y is $r(2)$ which is 2. The curve (3) is therefore that part of the parabola (4) which is above the line $y = 2$ and which is formed in the following manner. As t increases from $-\infty$ to 2, the tracing point moves from infinity along the upper portion of the parabola, through the vertex, and to the pseudo-rhamphoid cusp at (9, 2). A continued increase in t to $+\infty$ causes the tracing point to reverse its direction at (9, 2) and continue back through the vertex to infinity along the path upon which it arrived.

For the second example, take the curve whose parametric equations are

$$x = \cos^4 t, \quad y = \sin^4 t.$$

The reader will easily be able to show that this curve has, at each of the points (1, 0) and (0, 1), an infinite number of coincident pseudo-rhamphoid cusps and, furthermore, that its equations give an improper parametric representation of the parabola $(x-y)^2-2x-2y+1 = 0$.

NOTE ON DIRICHLET'S L-FUNCTIONS

By HANSRAJ GUPTA, *Government College, Hoshiarpur.*

[Received 9 November 1938]

1. Let $L(s)=\sum_{n=1}^{\infty}\chi(n)n^{-s},\ s>0;$

where $\chi(n)$ is a real non-principal character modulo k;

$$S_1(x)=\sum_{n\leqslant x}\chi(n),\ \text{and}\ S_m(x)=\sum_{n\leqslant x}S_{m-1}(n),\ m\geqslant 2.$$

Let m be the least positive integer (if any), such that

$$S_m(x)\geqslant 0,\ x\geqslant 1.$$

If m exists,* then S. Chowla† has shown that

$$L(s)>0,\ s>0.$$

At Chowla's suggestion, I have studied m for the primitive real characters modulo p, where p is a prime less than 500. I find that

$m=1$, when $p=3, 7, 11, 23, 31, 47, 59, 71, 79, 83$; $103, 131, 151, 167, 191, 199$; $239, 251, 263, 271$; $311, 359, 383$; $419, 431, 439, 479$.

$m=3$, when $p=53$; 211; 347.

It has not been possible to determine m when‡

$p=43, 67$; $163, 173, 197$; $283, 293$; $307, 317, 331, 379$; $443, 461, 463, 487, 499$.

For all other odd primes $p<500$, $m=2$.

It is noteworthy that for each of the primes for which m remains undetermined, most of the smaller primes are quadratic non-residues. The table at the end of this note gives

* H. A. Heilbronn has recently proved that m does not always exist. See *Acta Arithmetica*, 2 (1937), 212-13.

† S. Chowla, 'Note on Dirichlet's L-functions', *Acta Arithmetica* 1 (1935), 113-14.

‡ For these values of p, m does not appear to be small. It is possible that in these cases m does not exist.

the prime residues and non-residues less than 30 for each of the primes included in this list. R indicates the quadratic residues and N the non-residues. It may be of interest to note that if $m = 1$, then $p \equiv 3 \pmod 4$; but not conversely. This statement can be easily proved.

2. The Table.

p	2	3	5	7	11	13	17	19	23	29
43	N	N	N	N	R	R	R	N	R	N
67	N	N	N	N	N	N	R	R	R	R
163	N	N	N	N	N	N	N	N	N	N
173	N	N	N	N	N	R	N	N	R	R
197	N	N	N	R	N	N	N	R	R	R
283	N	N	N	R	R	R	N	N	R	R
293	N	N	N	N	N	N	R	N	N	N
307	N	N	N	R	R	N	R	R	N	N
317	N	N	N	R	R	N	N	N	R	N
331	N	N	R	N	N	N	R	R	N	N
379	N	N	R	N	N	N	N	R	R	N
443	N	R	N	N	N	R	R	N	N	N
461	N	N	R	N	N	N	R	R	R	N
463	R	N	N	N	N	N	R	N	N	R
487	R	N	N	N	N	N	N	R	N	R
499	N	N	R	N	N	N	N	N	N	R

REMARK ON THE THEORY OF BENDING OF PLATES OF VARIABLE THICKNESS III*

By ERIC REISSNER, *Massachusetts Institute of Technology.*

[Received 7 November 1938]

In this note we shall prove that for certain types of loading, the problem of bending and buckling of thin elastic rectangular plates of linearly varying stiffness admits very simple solutions, which are known from the theory of bending and buckling of bars and plates of constant thickness.

The differential equation for the deflection w of a thin elastic plate of varying stiffness $N(x, y)$ is*

$$\nabla^2(N\nabla^2 w) + (1-\nu)\left(2\frac{\partial^2 N}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial^2 N}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 N}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right) = p(x, y). \tag{1}$$

For linearly varying stiffness

$$N = N_0 + N_1 y; \tag{2}$$

this becomes

$$\nabla^2(N\nabla^2 w) = p. \tag{3}$$

If we assume

$$p(x, y) = (N_0 + N_1 y)P(x) \tag{4}$$

and

$$w(x, y) = W(x) \tag{5}$$

(3) becomes

$$(N_0 + N_1 y)W^{(iv)}(x) = (N_0 + N_1 y)P(x). \tag{6}$$

Thus the deflection has the same form as if the plate were of constant thickness and the load varying only in the x-direction. An example of such a problem is a plate, simply supported or built in along two opposite edges, in the direction of which the linear variation of the stiffness and the load occurs, and free along the

* I and II have appeared in *J. Math. Phys. M.I.T.* 16 (1937), 43-5 and in *Kong. Norske Vidensk. Selsk. Forh.* 10 (1937), 97-9.

two other edges. Howver the lateral contraction of the material along the free edges must not be taken into account.

In the same way can be treated the problem of buckling of this plate under the influence of a stress resultant in the plane of the plate

$$S_x = (N_0 + N_1 y) S = NS \tag{7}$$

which leads to the differential equation

$$\nabla^2 (N \nabla^2 w) = - NS \frac{\partial^2 w}{\partial x^2} \tag{8}$$

and is identical with the column buckling problem

$$W^{(\pi)} = -SW^{\pi} \tag{9}$$

if the previously indicated boundary conditions are prescribed.

A NOTE ON LAGUERRE'S FUNCTION

By RAM BEHARI, *University of Delhi.*

[Received 2 November 1938]

1. Taking a curve on a surface as the base curve, x, y, z, the coordinates of a point P on the base curve and X, Y, Z, the direction cosines of the normal to the surface at P, it has been proved in a previous paper* that Laguerre's function† $L'\left[=\frac{d}{ds}\left(\frac{1}{R}\right)+\frac{2}{\tau}\cdot\frac{1}{\gamma}\right]$, where R, τ, γ are the radii of normal curvature, geodesic torsion and geodesic curvature of the base curve, is exactly equivalent to the expression $\Sigma X''x'-\Sigma X'x''$, where accents indicate differentiation with respect to s.

The object of this note is to establish this equivalence by an entirely different method.

2. By the formulæ of Olinde Rodrigues, we have $\frac{dx}{dX}=\frac{dy}{dY}=\frac{dz}{dZ}=-r$ along the principal directions. Hence $rdX+dx=0, \ldots, \ldots$, or $X_1+Px_1=0$, $X_2+Qx_2=0$, etc., where P, Q are the principal curvatures.

The condition that the osculating quadric of the ruled surface formed by drawing normals to the surface along the curve on it be equilateral (*i.e.*, may have three mutually perpendicular generators) is

$$\Sigma X'x''-\Sigma X''x'=0‡. \tag{1}$$

Now

$$x'=x_1u'+x_2v',\ x''=x_{11}u'^2+2x_{12}u'v'+x_{22}v'^2+x_1u''+x_2v'',$$
$$X'=X_1u'+X_2v'=-Px_1u'-Qx_2v',$$
$$X''=-P(x_{11}u'^2+x_{12}u'v'+x_1u'')-Q(x_{21}u'v'+x_{22}v'^2+x_2v'')$$
$$-x_1u'(P_1u'+P_2v')-x_2v'(Q_1u'+Q_2v').$$

* R. Behari, *Jour. Ind. Math. Soc.*, 20 (1933), 218.

† Darboux, *Theorie generale des Surfaces*, t. II, (1915), p. 411; Blaschke *Vorlesungen über Differentialgeometrie*, Vol. I, p. 87.

‡ See R. Behari, *loc. cit.* p. 214.

$\therefore$ (1) becomes

$$\Sigma(Px_1u'+Qx_2v')\left\{(x_{11}u'^2+2x_{12}u'v'+x_{22}v'^2)+(x_1u''+x_2v'')\right\}$$

$$=\Sigma(x_1u'+x_2v')\Big[\Big\{P(x_{11}u'^2+x_{12}u'v')+Q(x_{21}u'v'+x_{22}v'^2)$$

$$+x_1u'(P_1u'+P_2v')+x_2v'(Q_1u'+Q_2v')\Big\}+Px_1u''+Qx_2v'')\Big]. \tag{2}$$

The terms containing u'', v'' reduce to

$$(\Sigma Qx_1x_2-\Sigma Px_1x_2)(v'u''-u'v'') \tag{3}$$

But as the line (X, Y, Z) is normal to the surface at P we have $\Sigma Xx_1=0$ and $\Sigma Xx_2=0$, and hence

$$\frac{\partial}{\partial u}(\Sigma Xx_2)=\frac{\partial}{\partial v}(\Sigma Xx_1), \text{ i.e., } \Sigma X_2x_1-\Sigma X_1x_2=0,$$

$\therefore$ $\Sigma Qx_1x_2-\Sigma Px_1x_2=0$. Hence (3) vanishes, and (2) becomes

$$u'^3(\Sigma x_1{}^2P_1)+v'^3(\Sigma x_2^2Q_2)+u'^2v'(-\Sigma Px_1x_{12}+\Sigma x_2x_{11}P-\Sigma Qx_2x_{11}$$
$$+\Sigma Qx_1x_{21}+\Sigma x_1^2P_2+\Sigma x_1x_2Q_1+\Sigma x_1x_2P_1)$$
$$+u'v'^2(-\Sigma Qx_2x_{12}+\Sigma Qx_1x_{22}-\Sigma Px_1x_{22}+\Sigma Px_2x_{12}+\Sigma x_1x_2Q_2$$
$$+\Sigma x_1x_2P_2+\Sigma x_2^2Q_1)=0.$$

But

$\Sigma x_1x_{11}=\frac{1}{2}E_1$, $\Sigma x_1x_{12}=\frac{1}{2}E_2$, $\Sigma x_1x_{22}=F_2-\frac{1}{2}G_1$, $\Sigma x_2x_{11}=F_1-\frac{1}{2}E_2$, $\Sigma x_2x_{12}=\frac{1}{2}G_1$, $\Sigma x_2x_{22}=\frac{1}{2}G_2$, where $\Sigma x_1^2=E$, $\Sigma x_1x_2=F$, $\Sigma x_2^2=G$.

Hence we get

$$u'^3(EP_1)+v'^3(GQ_2)+u'^2v'(-PE_2+QE_2+EP_2)$$
$$+u'v'^2(-QG_1+PG_1+GQ_1)=0, \text{ since } F=0,$$

or

$$\left\{u'^3(EP_1)+v'^3(GQ_2)+EP_2u'^2v'+GQ_1u'v'^2\right\}$$
$$+\frac{v'G_1}{\sqrt{EG}}\cdot\frac{1}{\tau}-\frac{u'E_2}{\sqrt{EG}}\cdot\frac{1}{\tau}=0, \tag{4}$$

where τ is the radius of geodesic torsion of the curve on the surface, since $\frac{1}{\tau}=u'v'\sqrt{EG}(P-Q)$.*

* See Forsyth, *Diff. Geo.*, p. 194.

Also, when the surface is referred to the lines of curvature, we have

$$\frac{1}{R}=Eu'^2P+Gv'^2Q=P\cos^2\theta+Q\sin^2\theta,$$

$$\frac{1}{\gamma'}=-\frac{E_2}{2EG^{\frac{1}{2}}},\quad \frac{1}{\gamma''}=\frac{G_1}{2GE^{\frac{1}{2}}},$$

where R is the radius of normal curvature of the curve, γ' and γ'' are the radii of geodesic curvatures of the curves $v=$ const. and $u=$ const., and θ is the angle between the curve and $v=$ const.*

Hence along the curve, we have

$$\frac{d}{ds}\left(\frac{1}{R}\right)=Eu'^2(P_1u'+P_2v')+Gv'^2(Q_1u'+Q_2v')$$
$$-2\,\theta'(P-Q)\cos\theta\sin\theta$$
$$=u'^3EP_1+v'^3GQ_2+EQ_2u'^2v'+GQ_1v'^2u'-\frac{2}{\tau}\theta'.$$

Hence (4) reduces to

$$\frac{d}{ds}\left(\frac{1}{R}\right)+\frac{2\theta'}{\tau}+\frac{v'G_1}{\sqrt{EG}}\frac{1}{\tau}-\frac{u'E_2}{\sqrt{EG}}\frac{1}{\tau}=0,$$

or

$$\frac{d}{ds}\left(\frac{1}{R}\right)+\frac{2}{\tau}\left(\theta'+\frac{\cos\theta}{\gamma'}+\frac{\sin\theta}{\gamma''}\right)=0,$$

since $u'\sqrt{E}=\cos\theta$, $v'\sqrt{G}=\sin\theta$,

or

$$\frac{d}{ds}\left(\frac{1}{R}\right)+\frac{2}{\tau}\cdot\frac{1}{\gamma}=0,$$

by Liouville's formula.†

Hence Laguerre's function

$$\frac{d}{ds}\left(\frac{1}{R}\right)+\frac{2}{\tau}\cdot\frac{1}{\gamma}$$

is equivalent to the expression $\Sigma X''x'-\Sigma X'x''$.

* See Eisenhart, *Diff. Geo.*, p. 134.
† Forsyth, *Diff. Geo*, p. 150.

ON WARING'S PROBLEM VIII. (WITH POLYNOMIAL SUMMANDS)*

By S. S. PILLAI, *Annamalainagar.*

[Received 25 January 1939]

1. *Introduction.* Let $G(f)$ denote the least value of s such that every large integer is the sum of s or fewer values of the polynomial $f(x)$ of degree n. Loo-Keng Hua under certain conditions and the author under milder conditions independently proved certain results which are roughly equivalent to the result

$$G(f) = O(n^3 \log n).$$

In this paper under still milder conditions certain results about $G(f)$ are proved, which are roughly equivalent to Vinogradow's result for $G(n)$. A lemma of Hua (Lemma 12) plays an important part in this paper.

2. *Statement of the results.* p always denotes a prime.

$f(x) \equiv a_0 x^n + \ldots + a_n$, where $a_0, \ldots, a_n$ are rational numbers in their lowest terms, and $a_0 > 0$.

M is the least common multiple of the denominators of $a_0, a_1, \ldots, a_n$.

$F(x) = Mf(x)$.

D is the greatest integer which divides $F(x)$ for all values of x.

$d = D/(D, M)$, where (a, b) denotes the greatest common divisor of a and b.

$a = a(p)$ and $t = t(p)$ are integers such that $F(x) \equiv a \pmod{p^t}$ is true for all values of x and suitable a; and $F(x) \equiv B \pmod{p^{t+1}}$ is not true for all x and any B.

$p^r = (a, p^t)$, where $r = r(p)$.

$A = \prod_p p^{t-r}$, where p runs through all primes.

* This is a slightly modified version of the paper submitted for the Ramanujan Memorial Prize (1937-1938), Madras University.

$\theta_r = \theta_r(p)$ is the highest power of p which divides $F^r(x)$ for all values of x, where $F^r(x)$ is the rth differential coefficient of $F(x)$.

$i_1 = \text{Max.}\frac{\theta_1 - \theta_r}{r-1}$, $r = 2, \ldots, n$.

$i = \text{Max}\{0, i_1\}$, when i_1 is an integer, or else,

$i = \text{Max}\{0, [i_1]+1\}$, where $[x]$ is the integral part of x.

$$b = b(p) = \begin{cases} \theta_1 + i + 2, \text{ when } p = 2. \\ \theta_1 + i + 1, \text{ when } p \geqslant 3. \end{cases}$$

$j = j(p) = \text{Max}\{b, t\}$.

$u = u(p)$ is the number of different residue classes in

$$F(x) \pmod{p^{t+1}}.$$

$V_1 = \text{Max}(p^{j-t}-1)/(u-1)+1$, when p runs through those primes for which $\theta_1 \geqslant 1$.

$V = V_1$, when V_1 is an integer, and $[V_1]+1$ otherwise.

$K = A + \text{Max}\{2n+1, V\}$.

*$G(f)$ is the least value of s such that every large number in the arithmetic progression $dN\{N = 1, 2, \ldots\}$ is the sum of s or fewer values of $f(x)$, where x takes only integral values.

Among the results proved in this paper, mention may be made here of the following.

THEOREM 2. *When* $n \geqslant 20$,

$$G(f) \leqslant \begin{cases} K, \textit{ when } V > 24n^4 \log n + 1 \\ K + 6n \log n + 2n \log\log n + 2n \log 12 + 2, \textit{ otherwise.} \end{cases}$$

THEOREM 3. *When* $n \geqslant 2$,

$$G(f) \leqslant \begin{cases} K, \textit{ when } V > n.2^n + 1, \\ K + 2n(n-1)\log 2 + 2, \textit{ otherwise.} \end{cases}$$

From Theorem 2, we get

* If a number is not a multiple of d, it is not representable as the sum of $f(x)$'s. If $d=1$ our definition agrees with the definition given in the introduction. So this is the better one.

THEOREM 6. *If $K = o\ (n \log n)$, then*

$$\overline{\lim} \frac{G(f)}{n \log n} \leqslant 6.$$

3. *The theory of the singular series.* The following further notations are used:

$$Q = \prod_p p^t$$

$$P = P(p) = p^j.$$

$$\rho = e^{2\pi i a/q},\ (a, q) = 1;\ S_{a,q} = \sum_{r=0}^{q-1} \rho^{F(r)};$$

$$B_{a,q} = \frac{1}{q} S_{a,q};\ A(q) = A(q, s, n) = \sum_a B_{a,q}^s \rho^{-N},$$

where a runs through all numbers not exceeding q and prime to q.

$$\mathfrak{S} = \mathfrak{S}(N) = \sum_{q=1}^{\infty} A(q);\ \chi_p = \chi_p(N) = \sum_{\lambda=0}^{\infty} A(p^\lambda).$$

$\Lambda(p^l) = \Lambda(p^l, N) =$ the number of solutions of

$$\sum_{\nu=1}^{s} F(h_\nu) \equiv N (\text{mod. } p^l),\ 0 \leqslant h_\nu < p^l. \qquad (1)$$

$\Delta(p^l) = \Delta(p^l, N) =$ the number of primitive solutions of (1), i.e. the number of solutions of (1) with the restriction that at least for one h_ν, $F'(h_\nu)$ is not a multiple of p^{θ_1+1}.

h' always denotes a number such that $F'(h')$ is not a multiple of p^{θ_1+1}.

LEMMA 1. *For $l \geqslant 0$,*

$$\sum_{\lambda=0}^{l} A(p^\lambda) = p^{-(s-1)l} \Lambda(p^l).$$

This is Lemma 4 in $\mathbf{R}(2)$.

LEMMA 2. *When $l \geqslant b$,*

$$F(x + yp^{l-\theta_1}) \equiv F(x) + yp^{l-\theta_1} F'(x) \,(\text{mod } p^{l+1}).$$

The general term in the expansion of $F(x + yp^{l-\theta_1})$ is $y^r p^{r(l-\theta_1)} F^r(x)/r!$. If m_r be the highest power of p which divides the general term for all x, then

$$m_r = r(l-\theta_1)+\theta_r - \sum_{s=1}^{\infty} [r/p^s]$$
$$> r(l-\theta_1)+\theta_r - r/(p-1).$$

(1) Let $p=2$, then
$$m_2 = 2(l-\theta_1)+\theta_2-1 \geqslant l+1, \text{ if } l \geqslant \theta_1+\theta_1-\theta_2+2.$$

When $r \geqslant 3$, $\quad m_r \geqslant l+1,$

if $\quad r(l-\theta_1)+\theta_r - r \geqslant l+1,$

or if $\quad l \geqslant \theta_1 + \dfrac{\theta_1-\theta_r}{r-1}+2.$

(2) Let $p \geqslant 3$, then
$$m_2 = 2(l-\theta_1)+\theta_2 \geqslant l+1, \text{ if } l \geqslant \theta_1+\theta_1-\theta_2+1.$$
$$m_3 \geqslant 3(l-\theta_1)+\theta_3-1 \geqslant l+1, \text{ if } l \geqslant \theta_1+\frac{\theta_1-\theta_3}{2}+1.$$

When $r \geqslant 4$,
$$m_r \geqslant r(l-\theta_1)+\theta_r - r/2 \geqslant l+1,$$

if $\quad l \geqslant \theta_1 + \dfrac{\theta_1-\theta_r}{r-1}+\dfrac{1+r/2}{r-1},$

or if $\quad l \geqslant \theta_1 + \dfrac{\theta_1-\theta_r}{r-1}+1.$

From the above, when $l \geqslant b$, $m_r \geqslant l+1$. Therefore the lemma is proved.

LEMMA 3. *When* $l \geqslant b$,
$$\Delta(p^{l+1}) = p^{s-1}\Delta(p^l).$$

Let
$$F(h') + \sum_{\nu=1}^{s-1} F(h_\nu) \equiv N \pmod{p^l}, \tag{2}$$

where $\quad h', h_1, \ldots, h_{s-1} \leqslant p^l - 1,$

and $F'(h')$ is not a multiple of p^{θ_1+1}

Now from Lemma 2,
$$F(h'+yp^{l-\theta_1}) + \sum_{\nu=1}^{s-1} F(h_\nu + y_\nu p^l) - N$$
$$\equiv F(h') + yp^{l-\theta_1}F'(h') + \sum_{\nu=1}^{s-1} F(h_\nu) - N \pmod{p^{l+1}}$$

$$=F(h')+\sum_{\nu=1}^{s-1}F(h_\nu)-N+p^l my,$$

$$= cp^l+p^l my, \text{ from } (2).$$

If $c+my\equiv 0 \pmod{p}$, then (3)

$$F(h'+yp^{l-\theta_1})+\sum_{1}^{s-1}F(h_\nu+y_\nu p^l)\equiv N \pmod{p^{l+1}}. \qquad (4)$$

Now, when $y_1,\ldots, y_{s-1}$ assume any value from 0 to $p-1$, $h_\nu+y_\nu p^l\leqslant p^{l+1}-1$. Since $(m, p)=1$, (3) is always solvable for y, and y is unique. Further, since $l\geqslant b$, it is easily seen that

$$F'(h'+yp^{l-\theta_1})\not\equiv 0 \pmod{p^{\theta_1+1}}.$$

So, corresponding to one primitive solution of (2), there are exactly p^{s-1} primitive solutions of (4). So the lemma is proved.

LEMMA 4. *When $l\geqslant b$,*

$$\Delta(p^l)=p^{(l-b)(s-1)}\Delta(P).$$

This follows from Lemma 3.

LEMMA 5. $\chi_p\geqslant\Delta(P)/P^{s-1}$.

From Lemma 1, when $l\geqslant b$,

$$\sum_{\lambda=0}^{l}A(p^\lambda)=\Delta(p^l)/p^{l(s-1)}\geqslant\Delta(p^l)/p^{l(s-1)}$$

$$\geqslant\Delta(P)/P^{s-1}, \text{ from Lemma 3.}$$

So
$$\chi_p=\sum_{\lambda=0}^{\infty}A(p^\lambda)=\lim_{l\to\infty}\sum_{\lambda=1}^{l}A(p^\lambda)$$

$$\geqslant\Delta(P)/P^{s-1}.$$

LEMMA 6. *Let $\alpha_1,\ldots,\alpha_m$ and $\beta_1,\ldots,\beta_n$ be two sets of different residue classes* (mod N)*; let $\gamma_1,\ldots,\gamma_l$ be all those different residue classes* (mod N) *which are representable as*

$$\alpha_i+\beta_j \quad (1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n).$$

Further let $d=\max\left\{N, \beta_1-\beta_s\right\}, 2\leqslant s\leqslant n$. Then $l\geqslant m+n-1$, provided $m+n-1\leqslant N/d$.

This is proved in **R**(3).

LEMMA 7. *If N is a multiple of D, there is a*

$$\psi = \psi(N) = \psi(N, p) \leqslant p^{t-r} - 1,$$

such that, when $s-1 \equiv \psi \pmod{p^{t-r}}$ *and*

$$s \geqslant (p^{j-t}-1)/(u-1)+1, \ \Delta(P) \geqslant 1.$$

Now $N - F(\overset{\circ}{h}{}')$ is a multiple of p^r. Further, since $(p, a/p_r) = 1$, there is a $\psi \leqslant p^{t-r}-1$, such that

$$\{N - F(h')\}/p^r \equiv \psi.(a/p^r) \pmod{p^{t-r}}$$
$$\equiv (s-1)(a/p^r) \pmod{p^{t-r}}, \qquad (5)$$

provided $\qquad s-1 \equiv \psi \pmod{p^{t-r}}$.

Therefore $\qquad N-F(h') \equiv (s-1)\, a \pmod{p^t} \qquad (6)$

$$\equiv (s-1)F(h) \pmod{p^t}.$$

Hence when $j = t$, the lemma immediately follows.

So we can assume that $j \geqslant t+1$.

Let $c_1, \ldots, c_u$ be the u different residue classes in

$$F(x) \pmod{p^{t+1}}.$$

Then $\qquad (p^{t+1}, c_m - c_s) = p^t, \ m \neq s.$

Choose x_m so that $F(x_m) \equiv c_m \pmod{p^{t+1}}$, $m = 1, \ldots, u$.

Let $\qquad E_m \equiv F(x_m) \pmod{P}, \ 1 \leqslant m \leqslant u.$

Then $\qquad (P, E_m - E_s) = p^t, \ m \neq s.$

So starting with $E_1, \ldots, E_u$ and applying Lemma 5 successively, we get that the number of residue classes (mod P) in $\beta_1 + \ldots + \beta_{s-1}$, where every $\beta = E_1, \ldots,$ or E_u, $\geqslant (s-1)u - (s-1) + 1$, provided $(s-1)u - (s-1) + 1 \leqslant p^{j-t}$ (7)

Hence we conclude that when

$$s-1 \geqslant (p^{j-t}-1)/(u-1).$$

(7) contains p^{j-t} residue classes, and they are

$$(s-1)a, \ (s-1)a + p^t, \ldots, \ (s-1)a + (p^{j-t}-1)p^t. \qquad (8)$$

From (6) when $s-1 \equiv \psi \pmod{p^{t-r}}$,

$$N - F(h') \equiv (s-1)a + mp^t \pmod{P},$$

where $\qquad 0 \leqslant m \leqslant p^{j-t} - 1.$

From this and (8), we conclude that

when $s-1 \equiv \psi \pmod{p^{t-r}}$ and $s-1 \geqslant (p^{j-t}-1)/(u-1)$,

$$N - F(h') \equiv \beta_1 + \ldots + \beta_{s-1} \pmod{P}$$

is solvable for β's in terms of E's. From this and the definition of E's, the lemma follows immediately.

Lemma 8. *If for a certain N_0 and a certain $s \geqslant V$, $\Delta(P, N_0) \geqslant 1$, then for every $N \equiv N_0 \pmod{Q}$, $\Delta(P, N) \geqslant 1$.*

Since from the previ us lemma $s \geqslant V$, $F(h') + F(x_1) + \ldots + F(x_{s-1})$, (with h' fixed) will contain the residue classes sa, $sa + p^t, \ldots$, $sa + (p^{j-t} - 1)p^t$, (mod P). By hypothesis, N_0 belongs to one of the above residue classes. Hence N also belongs to one of the above (mod P). So the lemma is established.

Lemma 9. *If $\theta_1 = t = 0$, then when $s \geqslant 2n+1$, $\Delta(P) \geqslant 1$.*

This is Lemma 16 in **R(2)**.

Lemma 10. *Corresponding to every N which is a multiple of D, there is one $l_0 = l_0(N)$ such that when $s - 1 \equiv l_0 \pmod{A}$ and $s \geqslant \max\{V, 2n+1\}$, $\Delta(P) \geqslant 1$ for every p.*

(1) $\theta_1 = t = 0$.

In this case, the result follows from Lemma 9.

(2) $\theta_1 \geqslant 1$, $t = 0$.

In this case, from Lemma 6, it follows that $\Delta(P) \geqslant 1$.

(3) $t \geqslant 1$.

It is possible to choose one $l_0 < A$, such that

$l_0 \equiv \psi(N, p) \pmod{p^{t-r}}$ for every p for which $t \geqslant 1$.

Hence the result follows from Lemma 6.

Lemma 11. *Corresponding to every N which is a multiple of D, there is one $l_0 < A$, such that when $s - 1 \equiv l_0 \pmod{A}$ and $s \geqslant \max\{V, 2n+1\}$, $\chi_p > 0$ for every p.*

For l_0 take the l_0 of the previous lemma. Then $\Delta(P) \geqslant 1$ for every p. So from Lemma 5,

$$\chi_p \geqslant \Delta(P)/P^{s-1} \geqslant 1/P^{s-1} > 0. \tag{9}$$

Lemma 12. *$S_{a,q} = O(q^{1-\frac{1}{n}+\varepsilon})$ for every $\varepsilon > 0$.* (10)

This is Lemma 3.1 in **R(1)**.

Lemma 13. *$A(q) = O(q^{1-s/n+\varepsilon})$ for every $\varepsilon > 0$.* (11)

This follows from the previous lemma.

LEMMA 14. *When* $s \geqslant 2n+1$, $\mathfrak{S} = \prod_p \chi_p$. (12)

$$\Big|\mathfrak{S} - \prod_{p \leqslant c} \chi_p\Big| \leqslant \sum_{q>c} |A(q)| = O\Big(\sum_{q>c} q^{1-(2n+1)/n+\varepsilon}\Big)$$

$$= O\Big(\sum q^{-1-1/2n}\Big) = O(c^{-1/2n}) \to 0 \text{ as } c \to \infty.$$

Hence the lemma follows.

LEMMA 15. *If* N *is a multiple of* D, *and* s *satisfies the conditions of Lemma* 10, *then* $\mathfrak{S} > c > 0$, *where* c *is a constant independent of* N.

The proof is exactly similar to that of Theorem II in **R(2)**. We have to apply Lemmas 13, 14 and 11.

LEMMA 16. *Under the hypotheses of Lemma* 10

$$\sum_{q \leqslant L} A(q) \geqslant c + O(L^{-1/2n}). \tag{13}$$

From Lemma 13

$$\Big|\sum_{q \leqslant L} A(q) - \mathfrak{S}\Big| \leqslant \sum_{q>L} |A(q)| = O(L^{-1/2n}).$$

From this and Lemma 15, the present one follows.

LEMMA 17. *If* N_0 *is a multiple of* D, *there is an* s *such that*

(1) $\max\{V, 2n+1\} \leqslant s \leqslant K$, *and*

(2) *for every* $N \equiv N_0 \pmod{Q}$, $\mathfrak{S}(N) > c > 0$, *where* c *is independent of* N.

From Lemma 11, the existence of s for which $\chi_p > c > 0$ follows. The rest follows from this, Lemma 8 and the proof of Lemma 15.

4. *Further notations.* k denotes a fixed positive integer.

Since $F(x)$ denotes only a single residue class (mod p^t), $F(x_1) + \ldots + F(x_{2k})$ denotes only a single residue class (mod Q) for all values of $x_1, \ldots, x_{2k}$. Let us denote this residue class by R_0. That is

$$F(x_1) + \ldots + F(x_{2k}) \equiv R_0 \pmod{Q}$$

identically.

Now for s we take the least value of s which satisfies the conditions of Lemma 17 when $N_0 = W - R_0$, where W is any fixed integer. Hence, from Lemma 17, we have when $N_2 \equiv N_0 \pmod{Q}$

$$\mathfrak{S}(N_2) > c > 0 \text{ for this } s = s(W, R_0, k). \tag{14}$$

Hereafter s stands for this particular value only. Further we take $N \equiv W \pmod{Q}$. (15)

Let K_0 be the sum of the moduli of the coefficients in $F(x)$.

N_1 is a fixed positive integer.

$F = [(N_1/K_0)^{1/n}/m]$, where m is large so that

$$m^n - m^{n-1} - 2.2^n - s \geqslant 1. \tag{16}$$

$N_1 - N_1 F^{-t_0} < N \leqslant N_1$ and N satisfies (15), where t_0 is a constant.

x_0 is a positive integer such that for all $x \geqslant x_0$, each of $F(x)$, $F'(x)$ and $F''(x)$ is positive. ξ runs through the numbers not exceeding F^{n-t_0}, which can be expressed as the sum of exactly k values of $F(x)$ with each $x \geqslant x_0$; and X denotes the number of such numbers $\leqslant F^{n-t_0}$.

$\sigma = n(1-1/n)^k$; $\tau = 2nK_0F^{n-\frac{1}{2}}$.

c is a positive constant, but it is different on different occasions.

$$T = \sum_{x=x_0}^{F} e^{2\pi i \alpha F(x)}; \quad T_1 = \sum_{x=x_0}^{K_1} e^{2\pi i \alpha F(x)},$$

where $K_1 = mK_0^{1/n}F$.

$$S = \sum_{\xi} e^{2\pi i \alpha \xi}; \quad T' = \sum_{x=x_0}^{F} e^{2\pi i a F(x)/q},$$

$$T_1' = \sum_{x=x_0}^{K_1} e^{2\pi i a F(x)/q}; \quad I_N = \int_0^1 T^{s-1} T_1 S^2 e^{-2\pi i \alpha N} d\alpha,$$

where for the interval of integration we can substitute $\tau^{-1} \leqslant \alpha \leqslant 1 - \tau^{-1}$. From the theory of Farey series, for every α in the last interval we have

$$\alpha = \frac{a}{q} + z; \ (a, q) = 1; \ 0 < q \leqslant \tau; \ |z| \leqslant 1/(q\tau). \tag{17}$$

We divide the interval of integration into two classes, putting in the first class the intervals with

$$\alpha=\frac{a}{q}+z;\ (a,q)=1;\ 0<q\leqslant F^{\frac{1}{2}};\ -\tau^{-1}\leqslant z\leqslant \tau^{-1} \quad (18)$$

and in the second class the remaining intervals. It is easy to see that the intervals of the first class do not overlap and that every α in the second class satisfies (17). Corresponding to this division we write

$$I_N=H_1+H_2. \quad (19)$$

$$\phi_1=\int_{x_0}^{F} e^{2\pi i z F(x)}dx;\ \phi_2=\int_{x_0}^{K_1} e^{2\pi i z F(x)}dx.$$

ϵ is a positive quantity which can be taken as small as we please.

5. *Some more Lemmas.*

LEMMA 18. *If* $0\leqslant f'(t)\leqslant \frac{1}{2}$; $f''(t)\geqslant c$ *in the interval* $g\leqslant t\leqslant h$, *then*

$$\sum_{t>g}^{h} e^{\pm 2\pi i f(t)}=\int_g^h e^{\pm 2\pi i f(t)}dt+O(1).$$

LEMMA 19. *If* λ *is real and positive and is not an integer, then*

$$\left|\sum_{x>b}^{h} e^{2\pi i\lambda x}\right|<1/2(\lambda),$$

where (x) *denotes the distance of* x *from the nearest integer.*

These two lemmas are well-known.

LEMMA 20. $X>c\,F^{n-\sigma-t_0}$.

This is Lemma 31 in **R(2)**.

LEMMA 21. *In the intervals of the second class, if*

$$S(F)=\sum_{x=1}^{F} e^{2\pi i a F(x)/q}=O(F^{1-\delta}),$$

then
$$\sum_{x=1}^{F} e^{2\pi i \alpha F(x)}=O(F^{1-\delta}).$$

For

$$\sum_{x=1}^{F} e^{2\pi i \alpha F(x)}=\sum e^{2\pi i\left(\frac{a}{q}+z\right)F(x)}$$

$$= \sum_{x=1}^{F} \left\{ S(x) - S(x-1) \right\} e^{2\pi i z F(x)}$$

$$= \sum_{x=1}^{F} S(x) \left\{ e^{2\pi i z F(x)} - e^{2\pi i z F(x-1)} \right\} + O\left\{ S(F) \right\}$$

$$= \sum S(x) e^{2\pi i z F(x)} \left\{ 1 - e^{2\pi i z \{F(x+1) - F(x)\}} \right\} + O(F^{1-\delta})$$

$$= \sum O\left\{ S(x) |z| x^{n-1} \right\} + O(F^{1-\delta})$$

$$= \sum O\left\{ x^{1-\delta} \times \frac{1}{q\tau} x^{n-1} \right\} + O(F^{1-\delta})$$

$$= \frac{1}{F^n} O\left(\sum x^{n-\delta} \right) + O(F^{1-\delta})$$

$$= O(F^{1-\delta}).$$

LEMMA 22. *Let* $\phi(x) = A_0 x^n + \dots + A_n$,

$$A_0 = \frac{a}{q} + \frac{\lambda}{q^2},\ |\lambda| \leqslant 1,\ (a, q) = 1, \text{ and}$$

$$F = \sum_{x=R+1}^{R+B} e^{2\pi i \phi(x)}$$

Then when $n \geqslant 20$, $F = O(B^{1-\delta})$ *where*

$\delta = 1/12 \cdot 2n^3 \log n$, *when* $c_1 B \leqslant q \leqslant c_2 B$,

$\delta = 1/20n^3 \log n$, *when* $c_2 B \leqslant q \leqslant c_3 B^{n-1-1/n}$,

and $\delta = 1/8n^3 \log (10n^2/c)$, *when* $c_3 B^{n-1-1/n} \leqslant q \leqslant c_4 B^{n-c}$,

where $c, c_1, \dots, c_4$ *are arbitrary positive constants.*

This is due to Vinogradow.*

LEMMA 23. *When* $F^{\frac{1}{2}} \leqslant q \leqslant F^{n-\frac{1}{2}}$ *and* $n \geqslant 20$,

$$T_1' = O(T') = O(F^{1-\delta_1}),$$

where $\delta_1 = 1/24n^3 \log n$.

PROOF: (1) $F^{\frac{1}{2}} \leqslant q \leqslant F$

$$|T'| \leqslant \left| \frac{F}{q} \sum_{x=0}^{q} e^{2\pi i a F(x)/q} \right| + \left| \sum_{x=1}^{q} 1 \right|,\ q_1 < q$$

$$= T_2 + T_3, \quad \text{(say)}.$$

* *Zentralblatt für Mathematik*, Band 13 Heft 9, p. 393.

From Lemma 12,

$$T_2 = O\left(\frac{F}{q} q^{1-1/n+\varepsilon}\right) = O(F^{1-\delta_1}).$$

From Lemma 22,

$T_3 = O(q_1)$ or $O(q^{1-1/12.2n^3 \log n})$ according as $q_1 >< F^{1-\delta_1}$.

$= O(F^{1-\delta_1})$.

So $\quad |T'| = O(F^{1-\delta_1})$. (20)

(2) $F < q \leqslant F^{n-1-1/n}$

From Lemma 22, $\quad T' = O(F^{1-\delta_1})$. (21)

(3) $F^{n-1-1/n} \leqslant q \leqslant F^{n-\frac{1}{2}}$.

In Lemma 22, put $c = \frac{1}{2}$.

Then $8 \log \frac{10n^2}{c} \geqslant 24 \log n$, for $n \geqslant 20$.

So $\quad T' = O(F^{1-\delta_1})$. (22)

From (20) and (21), the lemma follows.

LEMMA 24. *If* $\delta = 1/2^{n-1}$ *and* $\varepsilon > 0$, *then*

$$\sum_{x=R}^{R+B} e^{2\pi i a f(x)/q} = O(B^{\varepsilon} q^{\varepsilon})\left\{B^{1-\delta} + \frac{B}{q^{\delta}} + B^{1-n\delta} q^{\delta}\right\}.$$

This is a modified statement of Hilfsatz 6 in Landau's paper*.

LEMMA 25. *When* $n \geqslant 2$, $F^{\frac{1}{2}} \leqslant q \leqslant F^{n-\frac{1}{2}}$,

$$T'_1 = O(T') = O(F^{1-1/2^n}).$$

This is an immediate consequence of the previous lemma.

LEMMA 26. *If* $\quad Z = F$, *if* $|z| \leqslant F^{-n}$

and $\quad Z = |z|^{-n}$, *if* $|z| \geqslant F^{-n}$,

then $\quad \phi_2 = O(\phi_1) = O(Z)$.

This is proved in the footnote on page 189 in **R**(**2**). (Instead of f read F, and for a_0 put A_0.)

LEMMA 27. $\quad T'_1 = O(T') = O(q^{1-\delta})$

This is assumed.

6. *An Integral.*

6.1. *Asymptotic formula for* H_1.

Proceeding as in 3.3 in **R**(**2**), we get that

$$H_1 = Q_1 D_1 + O(X^2 F^{s-n-t_0}), \tag{23}$$

* 'Zum Waring's Problem' *Math. Zeit.*, Band 12, p. 236.

where Q_1 depends only on n, N_1 and x_0, and

$$D_1 = \sum_{q=1}^{F^{\frac{1}{2}}} \sum_{a} E_1 ;\ (a, q) = 1 \tag{24}$$

$$E_1 = B^s_{a,q} \Big(\sum_{\xi} e^{2\pi i a \xi / q} \Big)^2 e^{-2\pi i a N/q} \tag{25}$$

$$= O(X^2 q^{-s/n+\varepsilon}). \tag{26}$$

6.2. *Estimation of Q_1.* Again proceeding as in 3.4 in **R(2)**, we get that

$$\sum_N H_1 < c Q_1 X^2 F^{n-t_0} \text{ and } \sum_N I_N > c X^2 F^{s-t_0}. \tag{27}$$

In the intervals of the second class, by Lemma 19

$$\Big| \sum_N e^{-2\pi i \alpha N} \Big| = O\{ 1/(\alpha) \} = O(F^{n-\frac{1}{2}}), \tag{28}$$

and by Lemmas 27 and 21

$$T_1 = O(T) = O(F^{1-\delta}). \tag{29}$$

Hence $\displaystyle\sum_N H_2 = O\Big(F^{s-s\delta+n-\frac{1}{2}} \int_0^1 \sum_{\xi} \sum_{\xi'} e^{2\pi i(\xi-\xi')\alpha} d\alpha \Big)$

$$= O(F^{s+n-s\delta-\frac{1}{2}} X) = o(X^2 F^{s-t_0})$$

$$= o(X . F^{n-\sigma-t_0+s-t_0}) \text{ from Lemma 20,} \tag{30}$$

provided $\sigma + 2t_0 < s\delta + \frac{1}{2}$. (*A*)

Hence from (27) and (30), remembering that

$$I_N = H_1 + H_2$$

we get $Q_1 > c F^{s-n}$. (31)

6.3. *Estimation of D_1.* Again proceeding as in 3.5 in **R(2)**, we get

$$D_1 > c X^2 \tag{32}$$

provided $F^{1/4n} > c$. (*B*)

Hence from (23), (31) and (32), we have

$$H_1 > c X^2 F^{s-n} + O(X^2 F^{s-n-t_0})$$

$$> c X^2 F^{s-n} \tag{33}$$

provided $F^{t_0} > c$, (*C*)

6.4. *The fundamental result.* In the intervals of the second class, we have from Lemmas 27 and 21,

$$T_1 = O(T) = O(F^{1-\delta}).$$

So $$H_2 = O\left(F^{s-s\delta}\int_0^1 S^2 d\alpha \right) = O(F^{s-s\delta}X).$$

But from (33)

$$H_1 > cX^2F^{s-n} > cXF^{s-n+n-\sigma-t_0}$$
$$= cXF^{s-\sigma-t_0}.$$

Hence $H_2 = o(H_1)$, provided

$$s-s\delta < s-n-t_0, \text{ when } k=0, \qquad (D)$$

or $\sigma < s\delta - t_0$, when $k > 1$.

Hence $$I_N > cX^2F^{s-n} > 0, \qquad (34)$$

provided

(1) $\sigma < s\delta + \frac{1}{2} - 2t_0$;
(2) $F > c$
(3) $\sigma < s\delta - t_0$ when $k > 1$ and $s-s\delta < s-n-t_0$ when $k=0$.

When $s > n/\delta$, put $k=0$ and $t_0 = \frac{1}{2}\{ s-n/\delta \}$

and when $s \leqslant n/\delta$ put $k = \left[n \log \frac{n}{s\delta} \right] + 1$ and $t_0 = \delta/2n^2$.

Then conditions (1) and (3) are satisfied. Hence from (34) we arrive at the result that if $N > c$ and $k = \left[n \log \frac{n}{s\delta} \right] + 1$ when $s \leqslant n/\delta$, and $k=0$ when $s > n/\delta$, thenfor the particular s, $I_N > 0$.

But I_N is the number of representations of N in the form

$$N = F(x) + F(x_1) + \ldots + F(x_{s-1}) + \xi + \xi',$$

with $$x, x_1, \ldots, x_{s-1} \geqslant x_0.$$

The right side of the above is the sum of

$$1+s-1+2k = s+2k \text{ values of } F(x).$$

Therefore we arrive at the fundamental

LEMMA 28. *If $T' = O(F^{1-\delta})$ in the intervals of the second class, and $k = \left[n \log \frac{n}{s\delta} \right] + 1$ when $s \leqslant n/\delta$ and $k=0$ when $s > n/\delta$, then corresponding to every N which is a multiple of D, there is an $s \leqslant K$ such that N is the sum of $s+2k$ values of $F(x)$.*

7. *Proof of the main results.* Let R be a multiple of d so that $R = R_1 d$ where R_1 is an integer.

Then

$$MR = MR_1 d = MR_1 \frac{D}{(D, M)} = R_1 D \frac{M}{(M, D)} = N,$$

where N is a multiple of D.

So from Lemma 28

$$N = F(x_1) + \ldots + F(x_{s+2k}).$$

Therefore

$$R = \frac{N}{M} = \frac{F(x_1)}{M} + \ldots + \frac{F(x_{s+2k})}{M}$$
$$= f(x_1) + \ldots + f(x_{s+2k}).$$

Hence we arrive at

LEMMA 29. *Under the hypotheses of Lemma 28, corresponding to every N which is a multiple of d, there is an $s \leqslant K$, such that N is the sum of $s+2K$ values of $f(x)$.*

Now it can be easily verified that when $s \leqslant [n/\delta]+1$,

$$s+2\left[n \log \frac{n}{s\delta}\right]+2 \leqslant [n/\delta]+2.$$

So if $K > n/\delta + 1$, then

$$s+2\left[n \log \frac{n}{s\delta}\right]+2 \leqslant K.$$

Therefore we arrive at

THEOREM 1. *If $T' = O(F^{1-\delta})$ in the intervals of the second class, then*

$$G(f) \leqslant \begin{cases} K, \text{ when } V > 1+n/\delta \\ K+2[n \log (n/K\delta)]+2, \text{ when } V \leqslant 1+n/\delta. \end{cases}$$

When $n \geqslant 20$, from Lemma 23,

$$\delta = 1/24\, n^3 \log n.$$

So
$$n \log (n/K\delta) \leqslant n \log (24 n^4 \log n/K).$$

Hence we get

THEOREM 2. *When $n \geqslant 20$,*

$$G(f) \leqslant \begin{cases} K, \text{ when } V > 24n^4 \log n+1 \\ K+6n \log n+2n \log \log n+2n \log 12+2, \\ \qquad \text{when } V < 24n^4 \log n+1. \end{cases}$$

When $n \geqslant 2$, from Lemma 25,

$$\delta = 1/2^n.$$

Therefore we establish

THEOREM 3. *When* $n \geqslant 2$,

$$G(f) \leqslant \begin{cases} K, \text{ when } V > n.2^n+1 \\ K+2n(n-1)\log 2+2, \text{ when } V \leqslant 2^n+1. \end{cases}$$

course, we get also

THEOREM 4. *When* $n \geqslant 2$ *and* $K \leqslant n.2^n+1$

$$G(f) \leqslant K-2n \log K+2n^2 \log 2+2n \log n+2$$

and

THEOREM 5. *When* $n \geqslant 20$ *and* $K < 24n^4 \log n$,

$$G(f) \leqslant K-2n \log K+2n \log (24n^4 \log n)+2.$$

8. *Conclusion.* It is easy to see that when $f(x)=jx^n+1$, $G(f) \geqslant j$. Hence it is not possible to give the value of $G(f)$ in terms of n alone. The result proved in this paper is roughly equivalent to Vinogradow's result for $G(n)$. Further improvement in any one of these functions will have its effect on the other. The results so far proved about $G(n)$ and $G(f)$ are not final. If we can improve upon the result for $N_s(x)$, we can sharpen the results for $G(n)$ and $G(f)$. But the apparently simple function $N_s(x)$ defies all attempts to conquer it; no one has gone further than what the obvious elementary method leads to. Glory to the person who proves that

$$N_n(x) > x^{1-\varepsilon}.$$

REFERENCES

R(1) LOOKENG HUA: "On a generalized Waring's Problem", *Proc. London Math. Soc.* (2) 43 (1937).

R(2) S. S. PILLAI: "On Waring's Problem VI", *The Journal of the Annamalai University*, March 1936.

R(3) S S. PILLAI: "On the Addition of Residue classes", *Proc. Indian Acad. Sciences*, 7 (1938)

ON WARING'S PROBLEM IX

On universal Waring's problem with powers of primes

By S. S. PILLAI, *Annamalainagar.*

[Received 25 January 1939]

1. Vinogradow has proved not only the amazing result that every large number is the sum of a finite number of nth powers of primes, but has also found out a formula for its upper bound. Let $g_1(n)$ denote the least value of s such that every number from the beginning is the sum of s nth powers of primes or unity. Then it is likely that* when $n > n_0$

$$g_1(n) = 2^n + l - 2,$$

where

$$l = [(3/2)^n].$$

The object of this paper is to prepare the way for it. The main result proved is the following

THEOREM: *If* $1 \leqslant r \leqslant 2^n - 2l - 2$, *then when* $n \geqslant 20$, *every number not exceeding* N *is the sum of at most* $(2^n + l - 2)$ *nth powers of primes,*

where

$$3^n = l.2^n + r,$$

and

$$N = e^{(1.6)^n/6}.31^n. \tag{1}$$

It may be possible by following Vinogradow to show that every integer $\geqslant N$ is the sum of at most $(2^n + l - 2)$ nth powers of primes when $n > n_0$. If this is true, then we get

Conjecture. If $1 \leqslant r \leqslant 2^n - 2l - 2$, *then when* $n \geqslant n_0$,

$$g_1(n) = 2^n + l - 2.$$

But Vinogradow's paper on this topic is not accessible to the author.

* This conjecture was given by Hans Raj Gupta during a friendly discussion. This paper owes its origin to this conversation.

2. The following notations are used.

$$3^n = l.2^n + r, \quad 1 \leqslant r \leqslant 2^n - 1. \tag{2}$$

$$t = 2^n - r. \tag{3}$$

$$I = 2^n + l - 2. \tag{4}$$

$$N = e^{(1\cdot 6)^n/6}.31^n. \tag{5}$$

$$m = (\tfrac{3}{2})^n + (\tfrac{6}{3})^n + \ldots + (\tfrac{31}{29})^n. \tag{6}$$

p's and q's always denote primes. Throughout it is assumed that

$$n \geqslant 20. \tag{7}$$

$$l \leqslant r \leqslant 2^n - 2l - 2. \tag{8}$$

$$a = \begin{cases} [(2^{n-1} - l)/r] + 1 & \text{when } r \leqslant 2^{n-1} \\ [(2^{n-1} - 1)/t] & \text{when } r \geqslant 2^{n-1} \end{cases}. \tag{9}$$

u denotes the number of nth powers of primes and unity required to represent M.

3. To prove our result we want several lemmas.

LEMMA 1. *There will be at least one prime in* (α, β) *such that* $\alpha < p \leqslant \beta$, *if*

$$\alpha < \tfrac{5}{6}\beta - 2\beta^{\frac{1}{2}} - \frac{25}{16A \log 6}(\log \beta)^{\frac{1}{2}} - \frac{125}{24A}\log \beta - \frac{25}{6A},$$

where $\qquad A = \cdot 92129202.$

This is proved in the last chapter in Mathew's *Theory of Numbers*.

LEMMA 2. *When* $M \geqslant 25$, *there is at least one prime between* M *and* $\frac{5}{4} M$.

When $\beta = \frac{5}{4}\alpha$ and $\alpha \geqslant 10000$, it is easy to verify that the inequality in Lemma 1 is satisfied. Hence the lemma is proved when $M \geqslant 10000$. When $25 \leqslant N < 10000$, the lemma can be directly verified.

LEMMA 3. *Every integer* $\leqslant a.3^n$ *is the sum of at most* I *nth powers of primes.*

If M is any number $< a.3^n$,

then $\qquad M = b.3^n + d.2^n + s,$

where $\qquad 0 \leqslant b \leqslant a - 1, \; 0 < d \leqslant l \text{ and } s \leqslant 2^n - 1. \qquad (10)$

(1) Let $s \leqslant 2^n - l$.

Then $u \leqslant b+d+s \leqslant a-1+l+2^n-l$

$< 2^n+l-2;$

for from (9)

$$a \leqslant l-2.$$

(2) Let $s \geqslant 2^n - l$ and $b \geqslant 1$.

Then $M = (b-1)3^n + (d+l+1)2^n + s + r - 2^n,$

where $s+r-2^n \geqslant 0$; for $r \geqslant l$.

Hence

$$u \leqslant b-1+d+l+1+s+r-2^n$$
$$\leqslant l-2-1+l+l+1+2^n-1+2^n-2l-2-2^n$$
$$< I.$$

Now we have to consider the case when $b = 0$.

If $d \leqslant l-1$, then

$$u \leqslant l-1+2^n-1 = I.$$

Let $l.2^n \leqslant M < 3^n$. Now $M = l.2^n + M - l.2^n$.

So $u \leqslant l+M-l.2^n \leqslant l+3^n-l.2^n = l+r \leqslant I$.

The lemma is completely proved.

LEMMA 4.

*If** $a.3^n \leqslant M \leqslant a.3^n+2^n-1$, *then* $u \leqslant 2^{n-1}+2l+1$.

(1) Let $r \geqslant 2^{n-1}$. Then $2l+2 \leqslant t \leqslant 2^{n-1}$. (11)

Choose b so that $(b+1)t > s \geqslant bt$,

where $M = a.3^n + s$.

Then $s - bt \leqslant t-1$.

So $M = (a-b)3^n + b(l+1)2^n + s - bt$.

Hence $u = a-b+bl+b+s-bt$

$$\leqslant a+bl+t-1 \leqslant a(l+1)+t-1$$
$$\leqslant \frac{s}{t}(l+1)+t-1$$
$$\leqslant (2^n-1)(l+1)/t+t-1 \leqslant 2^{n-1}+2l+1$$

from (11).

* I got the hint to choose this interval from a paper of H. Gupta.

(2)[6] Let $r \leqslant 2^{n-1}$.

If $a.3^n \leqslant M \leqslant c.3^n + 2^{n-1} + l$,

then $u \leqslant 2^{n-1} + 2l$.

So we can assume that

$$M > a.3^n + 2^{n-1} + l.$$

Let $M = a.3^n + s$.

Then $2^{n-1} + l < s \leqslant 2^n - 1$.

Choose b so that

$$s + br \geqslant 2^n > s + (b-1)r.$$

Hence $s + br - 2^n \leqslant r - 1$.

$$M = (a-b)3^n + (bl+1)2^n + s + br - 2^n.$$

Therefore

$$\begin{aligned} u &\leqslant a - b + bl + 1 + s + br - 2^n \\ &\leqslant a + b(l-1) + 1 + r - 1 \leqslant al + r \\ &\leqslant \left(\frac{2^{n-1} - l}{r} + 1 \right) l + r \\ &< \frac{2^{n-1}.l}{r} + r + l \leqslant 2^{n-1} + 2l. \end{aligned}$$

So the proof of the lemma is completed.

LEMMA 5. *If* $u \leqslant s$ *when* $F \leqslant M \leqslant F + p^n$, *then when* $F \leqslant M \leqslant F + q^n$, $u \leqslant s + (q/p)^n$.

Let $F + p^n < M \leqslant F + q^n$

and $k = [(M-F)/p^n]$. (12)

Then $M = k.p^n + f$, where $F < f \leqslant F + p^n$.

Hence $u \leqslant k + s \leqslant (q/p)^n + s$.

So the lemma follows.

LEMMA 6. *If* $u \leqslant s$, *when* $F \leqslant M \leqslant F + 2^n$, *then when* $F \leqslant M \leqslant F + e^\delta.31^n$, $u \leqslant s + m + e\delta.(5/4)^n$.

Let $q_0 = 31$; $d = e^{1/n}$ and $b = 5/4$. (13)

Then from Lemma 2, we can choose primes $q_1, q_2, \ldots, q_\delta$ such that

$$\begin{aligned} dq_0 &< q_1 \leqslant dbq_0 \\ dq_1 &< q_2 \leqslant dbq_1 \\ &\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots \\ dq_{\delta-1} &< q_\delta \leqslant dbq_{\delta-1}. \end{aligned}$$

From the above inequalities, we get that

$$q_\delta > e^{\delta/n}.q_0 \text{ and } q_r/q_{r-1} \leqslant e^{1/n}.5/4,\ r = 1, \ldots, \delta. \qquad (14)$$

Now starting from the interval $(F, F+2^n)$ and applying Lemma 5 successively with 3, 5, 7, 11,..., 31, $q_1, \ldots, q_\delta$, we get

$$u \leqslant s+m+\left(\frac{q_1}{q_0}\right)^n+\ldots+\left(\frac{q_\delta}{q_{\delta-1}}\right)^n, \quad \text{if } F \leqslant M \leqslant F+q_\delta^n.$$

From this and the inequalities (14), the lemma follows.

4. Now we can prove our theorem.

THEOREM: *If* $l \leqslant r \leqslant 2^n-2i-2$, *then when* $M \leqslant N$ $\quad u \leqslant 2^n+l-2.$

In Lemma 6, let $F = a\,3^n$. Then, from Lemma 4.

$$s \leqslant 2^{n-1}+2l+1. \qquad (15)$$

In Lemma 6, put

$$\delta = [(1.6)^n/6]+1. \qquad (16)$$

Then since $n \geqslant 20$, from (15) and (16)

$$\begin{aligned}
&s+m+e\delta.(5/4)^n\\
&\leqslant 2^{n-1}+2l+1+e\left\{(1.6)^n/6+1\right\}(5/4)^n+m\\
&= 2^{n-1}+\frac{e}{6}.2^n+m+2l+e.(5/4)^n+1\\
&= 2^n+l-2-2^n(3-e)/6+e.(5/4)^n+m+l+3\\
&= 2^n+l-2-2^n\left\{\frac{3-e}{6}-\frac{e(5/4)^n+m+l+3}{2^n}\right\}\\
&< 2^n+l-2. \qquad (17)
\end{aligned}$$

Further

$$N \leqslant e^\delta.31^n. \qquad (18)$$

From (17), (18) and Lemma 3, the Theorem follows.

ON THE EXPANSION OF THE PARABOLIC CYLINDER FUNCTION IN A SERIES OF THE PRODUCT OF TWO PARABOLIC CYLINDER FUNCTIONS

By HARI SHANKER, *Delhi University.*

[Received 12 January 1939; in revised form 3 March 1939]

In an issue of the *Proceedings of the Benares Mathematical Society* (Vol. 6, 1924), I obtained an Addition Theorem for $D_n(z)$ in the form

$$D_n(x+y)=\frac{e^{\frac{1}{4}(x-y)^2}}{2^{n/2}}\sum_{m=0}^{n} {}_nC_m D_{n-m}(\sqrt{2}x)D_m(\sqrt{2}y). \qquad (1)$$

Various forms of Addition Theorems have been since then given by various writers like Messrs. Gorakh Prasad*, Dhar and Shastri†, and Dhar‡, but most of these forms like (1) are true only for integral values of n. In a paper published in the *Bulletin of the Calcutta Mathematical Society* (Vol. 26, 1934), Dr. Dhar, however, obtained by "the contour method" an expression for $D_n(x+y)$ in the form

$$D_n(x+y)=\Gamma(n+1)e^{-xy/2}\sum_{r=0}^{\infty}\frac{D_{n-r}(x)}{\Gamma(n-r+1)}\left\{\frac{D_r(y)}{r!}+\frac{1}{2}\frac{D_{r-2}(y)}{(r-2)!}+\frac{1}{2^2}\frac{D_{r-4}(y)}{(r-4)!\,2!}+\dots+\dots\right\}$$

which "holds for all values of n." The same method was employed by Dr. Gorakh Prasad*, when he expanded $D_n(u+v)$ in a series of ascending powers of v, the coefficients of various powers of v being expressed in terms of parabolic cylinder functions.

The object of the present paper is to find

(i) an Addition Theorem for $D_n(z)$, which may hold for all values of n and consequently may be regarded as an extended

* Gorakh Prasad: *Proc. B. M. Soc.* 4 (1922), also 7, 8 (1925-26).
† Dhar and Shastri: *Phil. Mag.* 18 (1934).
‡ Dhar: *Jour. Lond. Math. Soc.* 10 (1935).

and generalised form of the result (1), already obtained by me in 1924;

and (ii) to express $D_{n+m}(z)$ by a finite series involving the product of two parabolic cylinder functions.

1. *Addition Theorem.* The standard solution of Weber's Differential equation

$$\frac{d^2y}{dz^2}+(n+\tfrac{1}{2}-\tfrac{1}{4}z^2)y=0$$

is given by the relation*

$$e^{\frac{1}{4}z^2}D_n(z)=-\frac{\Gamma(n+1)}{2\pi i}\int_{\infty}^{+0}e^{-\frac{t^2}{2}-zt}(-t)^{-n-1}dt \qquad (2)$$

which in case $R(n)<0$, degenerates into the form

$$e^{\frac{1}{4}z^2}D_n(z)=\frac{1}{\Gamma(-n)}\int_0^{\infty}e^{-\frac{t^2}{2}-zt}t^{-n-1}dt. \qquad (2\text{-a})$$

Now putting $z=(ax+by)$ in (2), we obtain after a little rearrangement

$$e^{\frac{1}{4}(ax+by)^2}D_n(ax+by)$$

$$=-\frac{\Gamma(n+1)}{2\pi i}\int_{\infty}^{+0}e^{-axt-\frac{a^2t^2}{2r^2}}(-t)^{-n-1}.e^{-byt-\frac{b^2t^2}{2r^2}}\,dt.$$

where $r^2=a^2+b^2$.

Since†

$$e^{-\frac{t^2}{2}+zt-\frac{z^2}{4}}=\sum_{m=0}^{\infty}D_m(z)\frac{t^m}{m!}$$

we have

$$e^{\frac{1}{4}(ax+by)^2}D_n(ax+by)$$

$$=-\frac{\Gamma(n+1)}{2\pi i}e^{\frac{1}{4}r^2y^2}\int_{\infty}^{+0}e^{-axt-\frac{a^2t^2}{2r^2}}(-t)^{-n-1}\frac{\sum D_m(yr)\left(\frac{b}{r}\right)^m(-t)^m}{m!}dt$$

$$=-\frac{\Gamma(n+1)}{2\pi i}e^{\frac{1}{4}y^2r^2}\frac{\sum D_m(yr)\left(\frac{b}{r}\right)^m}{m!}\int_{\infty}^{+0}e^{-axt-\frac{a^2t^2}{2r^2}}(-t)^{m-n-1}dt$$

which after term by term integration, which is permissible, is equal to

* Whittaker and Watson: *Modern Analysis,* § 16.6.

† Hari Shanker: *Proc. B. M. Soc., loc. cit.*

$$\Gamma(n+1)e^{\frac{r^2}{4}(x^2+y^2)}\left(\frac{a}{r}\right)^n\sum_{m=0}^{\infty}\frac{D_{n-m}(rx)D_m(ry)\left(\frac{b}{a}\right)^m}{\Gamma(n-m+1)m!}.$$

Hence

$D_n(ax+by)$

$$=\Gamma(n+1)e^{\frac{1}{4}(bx-ay)^2}\left(\frac{a}{r}\right)^n\sum_{m=0}^{\infty}\frac{D_{n-m}(rx)D_m(ry)}{\Gamma(n-m+1)m!}\left(\frac{b}{a}\right)^m \tag{3}$$

In particular when n is a positive integer, by virtue of the well known result

$$\int_{\infty}^{+0} e^{-ux-\frac{1}{2}u^2}(-u)^{m-n-1}du = 0,$$

(m being integral and greater than n)

the result (3) reduces to a finite series of the form

$D_n(ax+by)$

$$=\Gamma(n+1)e^{\frac{1}{4}(bx-ay)^2}\left(\frac{a}{r}\right)^n\sum_{m=0}^{n}\frac{D_{n-m}(rx)D_m(ry)\left(\frac{b}{a}\right)^m}{\Gamma(n-m+1)m!} \tag{4}$$

which is a particular case of the very general result*

$$\frac{(a_1^2+\ldots+a_n^2)^{\frac{\mu}{2}}}{(1,\mu)}H_\mu\left(\frac{a_1x_1+\ldots a_nx_n}{\sqrt{a_1^2+\ldots+a_n^2}}\right)$$

$$=\sum_{m_1+\ldots+m_n=\mu}\frac{a_1^{m_1}\ldots a_n^{m_n}\,H_{m_1}(x_1)\ldots H_{m_n}(x_n)}{(1,m_1)\ldots(1,m_n)},$$

since $D_n(x)=e^{-\frac{1}{4}x^2}H_n(x)$, n being integral.

Again when $R(n)<0$, starting, similarly, from (2.a) we get

$D_n(ax+by)$

$$=\frac{e^{\frac{1}{4}(bx-ay)}\left(\frac{a}{r}\right)^n}{\Gamma(-n)}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(m-n)D_{n-m}(rx)D_m(ry)\left(\frac{b}{a}\right)^m}{m!}. \tag{5}$$

which is a degenerate form of the result (3) by virtue of the relation

$$\Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin n\pi}.$$

* Appell and Kampe de Feriet: *Fonctions Hypergeometriques et Hyperspherique, Polynome de Hermite* (1926), p. 345.

Hence we find that the result (3) is true for all values of n, and is consequently an extended and generalised form of the Addition Theorem 1, while the results (4) and (5) are mere deductions for particular values of n from this general result.

2. *Expression for* $D_{n+m}(z)$.

Since* $\frac{d^m}{dz^m}\left(e^{-\frac{1}{4}z^2}D_n(z)\right)=(-1)^m e^{-\frac{1}{4}z^2}D_{n+m}(z)$

and

$$\frac{d^m}{dz^m}\left\{ e^{\frac{1}{4}z^2}D_n(z) \right\}=n(n-1)(n-2)\ldots(n-m+1)D_{n-m}(z)$$

we have

$$\frac{d^m}{dz^m}\left[e^{-\frac{1}{4}z^2}D_n(z)\right] = \frac{d^m}{dz^m}\left\{ e^{-\frac{1}{4}z^2}D_0(z).e^{\frac{1}{4}z^2}D_n(z) \right\}$$

$$= (-1)^m e^{-\frac{1}{4}z^2}D_{n+m}(z)$$

because $D_0(z)=e^{-\frac{1}{4}z^2}$. Now by Leibnitz's Theorem we get

$$(-1)^m e^{-\frac{1}{4}z^2}D_{n+m}(z)$$

$$=(-1)^m\left\{ D_n(z)D_m(z)-mnD_{m-1}(z)D_{n-1}(z)+\ldots+ (-1)^m n(n-1)\ldots(n-m+1)D_{n-m}(z)D_0(z) \right\}$$

i.e.

$$e^{-\frac{1}{4}z^2}D_{n+m}(z) = \sum_{r=0}^{m} A_{n-r,\,m-r}D_{n-r}(z)D_{m-r}(z) \qquad (6)$$

where

$$A_{n-r,\,m-r}=\frac{(-1)^r n(n-1)\ldots(n-r+1)m(m-1)\ldots(m-r+1)}{r!},$$

which expresses a parabolic cylinder function of the order $(m+n)$ in terms of a finite series involving the product of two parabolic cylinder functions.

Putting $n=n-m$ in (6), we can express the result as

$$e^{-\frac{1}{4}z^2}D_n(z)=\sum_{r=0}^{m} A_{n-m-r,\,m-r}D_{n-m-r}(z)D_{m-r}(z), \qquad (7)$$

m being any arbitrary positive integer.

* Hari Shanker: *Proc. Ben. Math. Soc., loc. cit.*

Similarly, writing

$$\frac{d^m}{dz^m}\left\{ e^{\frac{1}{4}z^2} D_n(z) \right\} = \frac{d^m}{dz^m}\left[e^{-\frac{1}{4}z^2} D_n(z) e^{-\frac{1}{4}(iz)^2} D_0(iz) \right]$$

and applying Leibnitz's Theorem, we obtain

$$n(n-1)(n-2)\ldots(n-m+1)D_{n-m}(z)e^{\frac{1}{4}z^2}$$
$$=(-1)^m\left\{ D_n(z)D_m(iz)i^m + mD_{n+1}(z)D_{m-1}(iz)i^{m-1}+\ldots \right.$$
$$\left. +{}_mC_r\, i^{m-r}D_{n+r}(z)D_{m-r}(iz)+\ldots+D_{n+m}(z)D_0(iz) \right\}$$

i.e.

$$n(n-1)\ldots(n-r+1)D_{n-m}(z).e^{\frac{1}{4}z^2}$$
$$=(-1)^m \sum_{r=0}^{m} i^{m-r}{}_mC_r D_{n+r}(z)D_{m-r}(iz). \tag{8}$$

Again from the relation

$$D_n(iz)D_m(z) = e^{\frac{1}{4}z^2}D_n(iz)\,e^{-\frac{1}{4}z^2}D_m(z)$$

we have

$$\frac{d^p}{dz^p}\left[D_n(iz)D_m(z)\right]$$
$$=(-1)^p \sum_{r=0}^{p} i^{p-r}{}_pC_r D_{n+p-r}(iz)D_{m+r}(z). \tag{9}$$

and

$$\frac{d^p}{dz^p}\left[D_n(z)D_m(z)\right]$$
$$=(-1)^p\sum\frac{(-1)^r p(p-1)\ldots(p-r+1)m(m-1)\ldots(m-r+1)}{r!}$$
$$\times D_{n+p-r}(z)D_{m-r}(z)$$
$$=(-1)^p\sum A_{n+p-r,\,m-r}D_{n+p-r}(z)D_{m-r}(z) \tag{10}$$

where

$$A_{n+p-r,\,m-r}=\frac{(-1)^r p(p-1)\ldots(p-r+1)m(m-1)\ldots(m-r+1)}{r!}$$

ON THE PRINCIPLE OF CONSERVATION OF ENERGY IN THE RELATIVISTIC THEORY OF GRAVITATION

By D. N. MOGHE, *Research Scholar, Bombay University.*

[Received 20 February 1939]

1. *Introduction*:—The principle of conservation of energy has played a very important part in classical dynamics. But, as in almost a majority of cases where the motion of a system of particles or even of rigid bodies is studied, classical dynamics is generally replaced by relativity mechanics, it will be of some interest to examine this principle in the light of classical theory. Consider, for instance, the usual geodesic equations

$$\frac{d^2x^\alpha}{ds^2} = -\left\{ \mu\nu, \alpha \right\} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = F^\alpha \text{ (say)}, \tag{1}$$

where F^α is the acceleration vector in the classical sense; here F^α gives four components for $\alpha = 1, 2, 3, 4$, but it does not satisfy the conditions for a vector in general relativity. The velocity vector is given by

$$v^\mu = \frac{dx^\mu}{ds}, \tag{2}$$

so that the energy of the system can be expressed as

$$2\mathrm{T} = g_{\mu\nu}v^\mu v^\nu. \tag{3}$$

The usual Lagrange equations are then written as

$$\frac{d}{ds}\left(\frac{\partial \mathrm{T}}{\partial v^\mu}\right) - \frac{\partial \mathrm{T}}{\partial x^\mu} = 0. \tag{4}$$

The final form taken by (4) is[1]

$$v^\nu \frac{\partial B_\nu}{\partial v^\mu} - B_\mu = 0, \tag{5}$$

where

$$B_\mu = \tfrac{1}{2}\frac{\partial F^\alpha}{\partial v^\mu}\frac{\partial \mathrm{T}}{\partial v^\alpha} + \frac{\partial \mathrm{T}}{\partial x^\mu}, \tag{6}$$

that is,

$$\frac{\partial T}{\partial x^\mu}+\tfrac{1}{2}\frac{\partial F^\alpha}{\partial v^\mu}\frac{\partial T}{\partial v^\alpha}=0, \tag{7}$$

so that T is an integral of the system of partial differential equations (7). If we regard (2) as giving the motion, then an integral-invariant of the system is

$$\int\int\int\int\sqrt{-g}\ \delta x^1\delta x^2\delta x^3\delta x^4.^{(2)} \tag{8}$$

The condition that this should be an integral-invariant is that v^μ has to satisfy the equation

$$\frac{d\sqrt{-g}}{ds}+\sqrt{-g}\frac{\partial v^\mu}{\partial x^\mu}=0, \tag{9}$$

which reduces to

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}(v^\mu\sqrt{-g})=0. \tag{10}$$

Equation (10) gives us the permanence or conservation of the vector density $v^\mu\sqrt{-g}$.(3)

It has already been stated that the condition that Lagrange equations should define the motion of a dynamical system according to general relativity is that T must be an integral of (7). We can also write it as

$$v^\mu\frac{\partial T}{\partial x^\mu}+\tfrac{1}{2}v^\mu\frac{\partial F^\alpha}{\partial v^\mu}\frac{\partial T}{\partial v^\alpha}=0, \tag{11}$$

that is,

$$\frac{dT}{ds}=0. \tag{12}$$

We can find two interpretations of this result: (I) T must be constant along the same geodesic. This throws light on the principle of conservation of energy of classical dynamics, and in the light of this principle we can amplify the conclusion by saying that energy is conserved along the same geodesic, or, supposing that the geodesics are discrete, these tracks form the discrete energy levels of the dynamical system under consideration. (II) Lagrange equations define the geodesic motion of the fundamental particles in the universe. This goes to point out that Lagrange equations of classical dynamics pass over into geodesic equations of the general theory of relativity.

2. *Relativistic view of the conservation principle.* The vanishing of the covariant derivative of the energy-momentum tensor does not necessarily imply the conservation of energy or momentum in the strict classical sense. The conservation principle, if satisfied in the case of gravitational equations of general relativity, may require some additional term or terms to be included in the said equations, and Tolman has considered in a number of papers the necessity of such an additional term in the gravitational equations of general relativity so that the principle of conservation of energy may hold also for relativistic mechanics[4]. He has also suggested a new formulation of the gravitational equations of relativity to account for regular dissipation of (or a contribution to) the relativistic gravitational field. He has already suggested a constant rate of energy generation. The tensor equation for this case is

$$(T^{\mu\nu})_\nu = B^\mu = b_0 \frac{dx^\mu}{ds}, \tag{13}$$

where b_0 is an arbitrary constant, a parameter indicating in the general case the rate of generation or dissipation of energy per unit volume. The modification necessitates a new formulation of Einstein's equations so that

$$-8\pi T^{\mu\nu} = G^{\mu\nu} - \tfrac{1}{2} G g^{\mu\nu} + D^{\mu\nu} \tag{14}$$

The additional term $D^{\mu\nu}$ must be such that

$$(D^{\mu\nu})_\nu = -8\pi B^\mu. \tag{15}$$

As he himself admits Tolman has not arrived at a successful determination of $D^{\mu\nu}$ We suggest here

$$D^{\mu\nu} = -\varepsilon G g^{\mu\nu}, \tag{16}$$

so that

$$(D^{\mu\nu})_\nu = -\varepsilon g^{\mu\nu} \frac{\partial G}{\partial x^\nu} \tag{17}$$

ε is an arbitrary constant like b_0, and we have thus the rate of dissipation or generation of energy dependent upon both the parameter ε and the curvature G. Taking ε as of the first order, neglecting quantities of the second order, we may substitute $8\pi T$ for G in (17). Thus

$$(T^{\mu\nu})_\nu = -\varepsilon g^{\mu\nu} \frac{\partial T}{\partial x^\nu}. \tag{18}$$

The energy equation is given by $\mu = 4$.

We shall now consider this new equation in connection with two problems, viz., (i) The non-static universe, and (ii) the internal field of a perfect fluid.

3. *The Non-Static Universe.* For the line element

$$ds^2=-R^2[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta\, d\phi^2)]+dt^2 \tag{19}$$

$$T_1^1=T_2^2=T_3^3=-p, \text{ and } T_4^4=\rho, \tag{20}$$

of the Friedmann-Lemaitre solution; the equation of conservation of energy is

$$\rho+\frac{3\dot{R}}{R}(\rho+p)=0, \tag{21}$$

where a dot denotes a differentiation with regard to time. When the energy is not conserved and (18) is satisfied,

$$\rho+\frac{3\dot{R}}{R}(\rho+p)=\varepsilon(\dot{\rho}-3\dot{p}). \tag{22}$$

If the pressure is due to radiation only let σ be the proper density, σ_1 the density of radiation, so that

$$\sigma=\rho-3p, \quad \sigma_1=3p. \tag{23}$$

Thus the rate at which the energy changes is measured by the rate at which the proper density is changing. The dissipation of matter into radiation serves as a measure of the dissipation of energy in a non-conserving field. It may be noticed that ε is an arbitrary parameter not yet fixed, its sign also is undetermined. Whether there is a loss or gain of energy per unit time depends upon the sign of ε. It is interesting to notice that the condition for a universe of conserved energy is that the average proper density must remain constant. If $\rho-3p=\sigma$ is constant, then

$$\rho+\frac{\dot{R}}{R}(3\rho+\rho-\sigma)=0, \tag{24}$$

which gives

$$R^4(4\rho-\sigma)=\text{constant}. \tag{25}$$

We have now

$$G^{\mu\nu}-\lambda g^{\mu\nu}=-8\pi(T^{\mu\nu}-\tfrac{1}{2}Tg^{\mu\nu})-4\varepsilon g_0^{\mu\nu}(\lambda+2\pi T), \tag{26}$$

with the condition that $\varepsilon=0$, when there is no material distribution, and

$$G^{\mu\nu}-\lambda g^{\mu\nu}=0 \tag{27}$$

for empty space. From (25), ρ and p become known and the march of R with time becomes determinate from an easy differ-

ential equation given by (26). Two cases arise in several other cognate discussions. In one case R has a stationary value and in the other R is a monotonic function of time.

4. *A Non-Conservative Fluid Sphere.* Following the notation in Tolman's treatise on relativity, the line element of the interior solution for a fluid sphere of constant density (ρ_{00}) runs as follows:—

$$ds^2 = -\frac{dr^2}{(1-r^2/R^2)} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + [A - B\sqrt{(1-r^2/R^2)}]dt^2, \tag{28}$$

where

$$R^2 = 3/8\pi\rho,\ A = \tfrac{3}{2}\sqrt{(1-r_1^2/R^2)},\ B = \tfrac{1}{2},\ m = \frac{4\pi}{3}\rho r_1^3, \tag{29}$$

m being the so-called "mass" of the sphere and r_1 the radius. This line-element has to be slightly modified if equations for a non-conservative field are to hold good. If as a result of the field being slightly non-conservative, A and R are generalized as slowly varying functions of time, the conditions of isotropy, $T_1^1 = T_2^2 = T_3^3$ then give

$$\ddot{R} = (3\dot{R}^2/R)/(1-r^2/R^2) + \frac{\dot{R}[\dot{A} - Br^2\dot{R}/R^3)/\sqrt{(1-r^2/R^2)}]}{A - B\sqrt{(1-r^2/R^2)}} \tag{30}$$

For $\dot{A}$ we may write $\tfrac{3}{2}\,\dfrac{r_1^2\dot{R}}{R^3}(1-r_1^2/R^2)^{-\frac{1}{2}}$.

The energy equation $(T^{4\nu})_\nu = \varepsilon g^{44}\dfrac{\partial T}{\partial t}$ yields

$$8\pi\varepsilon\dot{\sigma} = \frac{\dot{R}/R^3[(1-2r^2/R^2)(1-r^2/R^2)^{-\frac{1}{2}} - 3(1-5r_1^2/2R^2)(1-r_1^2/R^2)^{-\frac{1}{2}}]}{\tfrac{3}{2}\sqrt{(1-r_1^2/R^2)} - \tfrac{1}{2}\sqrt{(1-r^2/R^2)}}, \tag{31}$$

where σ is the proper density.

If r_1/R is neglected

$$\ddot{R} = 3\dot{R}^2/R,\ \text{giving}\ R = (a+2bt)^{-\frac{1}{2}}, \tag{32}$$

where a and b are constants of integration. Hence we can write

$$8\pi\varepsilon\dot{\sigma} = -2\dot{R}/R^3 = 2b, \tag{33}$$

$$m = r_1^3/2R^2 = \frac{r_1^3}{2}\cdot(a+2bt), \tag{34}$$

$$\dot{m} = 6r_1^3 = 4\pi\varepsilon\dot{\sigma}r_1^3. \tag{35}$$

Thus the rate at which the mass of the fluid sphere is dissipated away depends upon the rate at which the proper density decreases, a result quite in agreement with expectation.

5. *Appendix: The exterior solution.* When the cosmological term is present in the gravitational equations we have

$$-8\pi T^{\mu\nu} = G^{\mu\nu} + (\lambda - \tfrac{1}{2}G) g^{\mu\nu}. \qquad (36)$$

Then for the field to be conservative we insert, as before, an additional term in (36) so that

$$-8\pi T^{\mu\nu} = G^{\mu\nu} + \left\{ \lambda - (\varepsilon + \tfrac{1}{2}) G \right\} g^{\mu\nu}. \qquad (37)$$

To the second order of small quantities the conservation law is

$$(T^{\mu\nu})_{\nu} = -\varepsilon g^{\mu\nu} \frac{\partial T}{\partial x^{\nu}}, \qquad (38)$$

as before. For the exterior solution of Schwarzchild we have

$$T^{\mu\nu} = 0 \text{ and } T = 0. \qquad (39)$$

Therefore ε cannot be determined from (38). It should be remembered that $\lambda \neq \varepsilon G$, as then there will be no cosmological term present in the gravitational equations.

It is a great pleasure to record my gratitude to Prof. V. V. Narlikar, under whose advice some sections of this paper were worked out.

REFERENCES

(1) E. T. WHITTAKER: *Analytical Dynamics*, pp. 285-6.

(2) ——, *loc. cit*, p. 283.

(3) EDDINGTON: *Math Theory of Rel*, p. 115.

(4) TOLMAN: *Proc. Nat. Acad. Sci.* 20, 6, 379-383; 20, 7, 437-439; 20, 7, 439-444.

ON THE THEORY OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE PARABOLIC TYPE

By S. MINAKSHISUNDARAM, *Madras University.*

[Received 3 March 1939]

1. The object of this paper is to study the non-linear equations

$$\frac{\partial}{\partial x}\left\{p(x)\frac{\partial u}{\partial x}\right\}-\frac{\partial u}{\partial t}=\sum_{r=2}^{\infty} p_r(x,t)u^r \tag{1}$$

$$\frac{\partial}{\partial x}\left\{p(x)\frac{\partial u}{\partial x}\right\}-\frac{\partial u}{\partial t}=\sum_{r,s=1}^{\infty} p_{r,s}(x,t)u^r\left(\frac{\partial u}{\partial x}\right)^s \tag{2}$$

for the second boundary conditions

$$\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(\pi,t)=0,\ t\geqslant 0 \tag{3}$$

$$u(x,0)=f(x) \tag{4}$$

where $p(x)$ is an essentially positive function defined in $(0,\pi)$ and where $p(x)$, $p'(x)$, $p''(x)$ are continuous and bounded in the interval $(0,\pi)$.

The equations (1) and (2) for the first boundary conditions

$$u(0,t)=u(\pi,t)=0,\ t\geqslant 0 \tag{3'}$$

$$u(x,0)=f(x) \tag{4}$$

were studied by Prof. M. R. Siddiqi* in a series of papers, and he proved that one and only one solution of (1) and (2) exists satisfying (3′) and (4), which can be expanded in a series of Sturm-Liouville eigen-functions. Following the method of Prof. Siddiqi, it is shown here, that a solution of (1) and (2) is still possible in the second boundary value problem. It is of interest to note the difference in the result. Whereas it can be shown that in the case of the first boundary value problem there exist solutions

* (1) *Math. Zeit.* 35 (1932), 461-484. (2) *Ibid.* 40 (1935), 484-495. (3) *Comptes Rendus* 203 (1936), 1120.

for all values of t, in the case of the second boundary value problem, solution is possible only for small values of t.

The paper then concludes with a few remarks on equations in more than two independent variables.

2. For the sake of simplicity we consider the equations

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = u^2 \tag{5}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} \tag{6}$$

for the boundary conditions (3) and (4). The general equations (1) and (2) can be similarly studied in view of the discussions in the papers of Prof. Siddiqi. We shall expand $f(x)$ in a cosine series

$$f(x) \sim \sum_{n=0}^{\infty} c_n \cos nx\,;\; c_0 = \frac{1}{\pi}\int_0^{\pi} f(t)\,dt,$$

$$c_n = \frac{2}{\pi}\int_0^{\pi} f(t) \cos nt\, dt, \tag{7}$$

and solve (5) and (6) in the form

$$u(x, t) = \sum_{n=0}^{\infty} u_n(t) \cos nx\,;\; u_0(t) = \frac{1}{\pi}\int_0^{\pi} u(x, t)\,dx,$$

$$u_n(t) = \frac{2}{\pi}\int_0^{\pi} u \cos nx\, dx. \tag{8}$$

2.1. Consider first the equation (5). If we write

$$u^2(x, t) = \sum_{n=0}^{\infty} z_n(t) \cos nx, \tag{9}$$

then

$$\begin{aligned} z_0(t) &= \frac{1}{\pi}\int_0^{\pi} u^2(x, t)\,dx \\ &= \sum_{k, l=0}^{\infty} u_k u_l \cdot \frac{1}{\pi}\int_0^{\pi} \cos kx \cos lx\, dx \\ &= \sum_{k=0}^{\infty} u_k^2. \end{aligned} \tag{10.1}$$

$$z_n(t)=\frac{2}{\pi}\int_0^\pi u^2 \cos nx\ dx$$

$$=\sum_{k,\,l=0}^{\infty} u_k u_l \cdot \frac{2}{\pi}\int_0^\pi \cos kx \cos lx \cos nx\ dx$$

$$=\sum_{k,\,l=0}^{\infty} a_{k,\,l}^n u_k u_l, \text{ say,}$$

$$=2u_0u_n+\sum_{k,\,l=1}^{\infty} a_{kl}^n u_k u_l, \tag{10.2}$$

for

$$\left.\begin{aligned} a_{00}^n &= \frac{2}{\pi}\int_0^\pi \cos nx dx = 0 \\ a_{0l}^n &= \frac{2}{\pi}\int_0^\pi \cos lx \ \cos nx dx = \begin{cases} 0 \text{ if } l \neq n \\ 1 \text{ if } l = n \end{cases} \\ a_{k0}^n &= \frac{2}{\pi}\int_0^\pi \cos kx \cos nx dx = \begin{cases} 0 \text{ if } k \neq n \\ 1 \text{ if } k = n \end{cases} \end{aligned}\right\} \tag{11}$$

If we substitute (8) in (5) and equate the Fourier coefficients of both sides, we have

$$\left.\begin{aligned} &\frac{du_0}{dt} = -z_0(t) \\ &\frac{du_n}{dt} + n^2 u_n = -z_n(t)\ ;\ n = 1, 2, \ldots \end{aligned}\right\}, \tag{12}$$

which is equivalent to

$$\left.\begin{aligned} u_0(t) &= c_0 - \int_0^t z_0(s)ds \\ u_n(t) &= c_n e^{-n^2 t} - \int_0^t e^{-n^2(t-s)} z_n(s)ds \end{aligned}\right\} \tag{13}$$

Now set

$$\left.\begin{aligned} &v_0 = u_0\ ;\ \gamma_0 = c_0 \\ &v_n = n^2 u_n\ ;\ \gamma_n = n^2 c_n\ ;\ n = 1, 2, \ldots \end{aligned}\right\}. \tag{14}$$

We have, from (13),

$$v_0 = \gamma_0 - \int_0^t \left\{ v_0^2(s) + \sum_{k=1}^{\infty} \frac{v_k^2(s)}{k^4} \right\} ds$$

$$v_n = \gamma_n e^{-n^2 t} - n^2 \int_0^t e^{-n^2(t-s)} \left\{ \frac{2v_0 v_n}{n^2} + \sum_{k,\,l}^{1,\ldots,\infty} \frac{a_{kl}^n}{k^2 l^2} v_k v_l \right\} ds. \qquad (15)$$

The equations (15), can also be written in the form

$$v_0 = \gamma_0 - \int_0^t \sum_{k,\,l=0}^{\infty} d_{kl}^0 v_k v_l \, ds$$

$$v_n = \gamma_n e^{-n^2 t} - n^2 \int_0^t e^{-n^2(t-s)} \sum_{k,\,l=0}^{\infty} d_{kl}^n v_k v_l \, ds, \qquad (16)$$

where

$$d_{kl}^n = \begin{cases} \dfrac{a_{kl}^n}{k^2 l^2} \text{ if } k,\, l > 0 \\[2mm] \dfrac{a_{kl}^n}{l^2} \text{ if } k = 0,\, l > 0 \\[2mm] \dfrac{a_{kl}^n}{k^2} \text{ if } k > 0,\, l = 0 \\[2mm] a_{kl}^n \text{ if } k = l = 0. \end{cases} \qquad (17)$$

We shall solve the infinite system of non-linear integral equations (16) by successive approximations. It can be shown that the series $\sum_{n=0}^{\infty} |d_{k,l}^n|$ is uniformly convergent* for all values of k and l, and let us suppose that

$$\sum_{n=0}^{\infty} |d_{kl}^n| \leqslant A \qquad (18)$$

and

$$\sum_{n=0}^{\infty} |\gamma_n| = c, \qquad (19)$$

where A and c are positive constants. We shall show that if

$$4Ac < 1 \qquad (20)$$

* See, for example, the lemma on page 486 of the second paper of Prof. Siddiqi already quoted.

and if

$$t < \frac{1}{4Ac}; \tag{21}$$

we can solve (16) uniquely such that $\Sigma|v_n|$ is convergent uniformly. In (16), set

$$\left.\begin{aligned} v_0^0 &= \gamma_0 \\ v_0^m &= \gamma_0 - \int_0^t \sum_{k,l} d_{kl}^0 v_k^{m-1} v_l^{m-1}\, ds \\ v_n^0 &= \gamma_0 e^{-n^2 t} \\ v_n^m &= \gamma_n c^{-n^2 t} - n^2 \int_0^t e^{-n^2(t-s)} \sum_{k,l} d_{kl}^n v_k^{m-1} v_l^{m-1}\, ds \end{aligned}\right\} \tag{22}$$

and for brevity put

$$z_n^m = \sum_{k,l} d_{kl}^n v_k^m v_l^m \tag{23}$$

assuming the convergence of the double series on the right which will be presently proved. We have now from (22)

$$\sum_{n=0}^{\infty} |v_n^m| \leqslant \sum_{n=0}^{\infty} |\gamma_n| + \int_0^t |z_0^{m-1}(s)|ds + \sum_{n=1}^{\infty} \text{Max } |z_n^{m-1}(t)|$$

$$\leqslant c + \text{Max}(t, 1). \text{ Max} \sum_{n=0}^{\infty} |z_n^{m-1}(t)|^*. \tag{24}$$

Now

$$\text{Max} \sum_{n=0}^{\infty} |z_n^{m-1}(s)| = \text{Max} \sum_{n=0}^{\infty} \left| \sum_{k,l} d_{kl}^n v_k^{m-1} v_l^{m-1} \right|$$

$$\leqslant \text{Max} \sum_{n} \sum_{k,l} |d_{kl}^n|\ |v_k^{m-1}|\ |v_l^{m-1}|$$

$$\leqslant A\left(\text{Max} \sum_{k=0}^{\infty} |v_k^{m-1}\right)^2 \tag{25}$$

* Without any danger of confusion we will write throughout $\text{Max} \sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} \text{Max } |a_n|$, where a_n's are functions.

If we write

$$\operatorname{Max} \sum_{k=0}^{\infty} |v_k^m| = \varphi_m \tag{26}$$

we observe that

$$\varphi_m \leqslant c + A.\ \operatorname{Max}(t,\ 1)\varphi_{m-1}^2. \tag{27}$$

If we put $m = 1$, since

$$\varphi_0 = \sum_{n=1}^{\infty} |\gamma_n e^{-n^2 t}| \leqslant \sum |\gamma_n| = c$$

then

$$\varphi_1 \leqslant c + Ac^2 \operatorname{Max}(t,\ 1).$$
$$< 2c \text{ if } \operatorname{Max}(t,\ 1) < \frac{1}{Ac}.$$

Generally, if $\varphi_{m-1} \leqslant 2c$.

$$\varphi_m \leqslant c + 4Ac^2 \operatorname{Max}(t,\ 1) \tag{28}$$
$$< 2c \text{ if } \operatorname{Max}(t,\ 1) < \frac{1}{4Ac}.$$

The condition

$$\operatorname{Max}(t,\ 1) < \frac{1}{4Ac} \tag{29}$$

is equivalent to (20) and (21). Assuming (29), we shall show that the double series

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |v_n^{m+1} - v_n^m| \tag{30}$$

is uniformly convergent. We have, in fact, as in (24)

$$\sum_{n=0}^{\infty} |v_n^{m+1} - v_n^m| \leqslant \operatorname{Max}(t,\ 1) \sum_{n=0}^{\infty} |z_n^m - z_n^{m-1}|$$

$$\leqslant \operatorname{Max}(t,\ 1) \sum_{n=0}^{\infty} \sum |d_{kl}^n|\ |v_k^m v_l^m - v_k^{m-1} v_l^{m-1}$$

$$\leqslant \operatorname{Max}(t,\ 1) \sum_{n=0}^{\infty} \sum_{k,l} |d_{kl}^n| \Big\{ |v_k^m|\ |v_l^m - v_l^{m-1}|$$
$$+ |v_l^{m-1}|\ |v_k^m - v_k^{m-1}| \Big\}$$

$$\leqslant \text{Max }(t, 1). A.2.2c. \sum_{k=0}^{\infty} |v_k^m - v_k^{m-1}|$$

$$= 4Ac. \text{ Max }(t, 1) \sum_{n=0}^{\infty} |v_n^m - v_n^{m-1}|$$

$$< \{ 4Ac. \text{ Max }(t, 1) \}^m \sum_{n=0}^{\infty} |v_n^1 - v_n^0|$$

$$< Ac.^2 \{ 4Ac. \text{ Max }(t, 1) \}^m \tag{31}$$

On account of (29), we observe that the series (30) is uniformly convergent, which implies that the limits

$$v_n(t) = \lim_{m \to \infty} v_n^m(t), \; n = 0, 1, 2, \ldots \tag{32}$$

exist, and that the $v_n(t)$'s are continuous. Further, for t satisfying (21)

$$\sum_{n=0}^{\infty} |v_n(t)| < 2c. \tag{33}$$

That the functions $v_n(t)$ are the unique solutions of (16) and that

$u(x, t) = \sum_{n=0}^{\infty} u_n(t) \cos nx, \; u_n = \frac{v_n}{n^2}$ is the required solution of (5)

can be proved, as is done by Prof. Siddiqi.

2 2. With slight alterations in the above treatment we can solve the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x}, \tag{6}$$

for the same boundary conditions. We have to assume the convergence of $\Sigma n^3 |c_n|$ and write

$$c_0 + \sum_{n=1}^{\infty} |n^3 c_n| = c' \tag{34}$$

instead of (19). For (9) we write

$$u \frac{\partial u}{\partial x} = \sum_{n=0}^{\infty} z'_n(t) \cos nx \tag{35}$$

where

$$z'_n(t) = \sum_{k,l}^{0,\ldots,\infty} a'^n_{kl} u_k u_l \tag{36}$$

$$a'^n_{kl} = \frac{2l}{\pi}\int_0^\pi \cos kx \sin lx \cos nx \, dx, \text{ if } n \neq 0 \tag{37.1}$$

and

$$a'^0_{kl} = \frac{l}{\pi}\int_0^\pi \cos kx \sin lx \, dx. \tag{37.2}$$

Then in the place of (18), we have to prove the uniform convergence of

$|d'^0_{kl}| + \sum_{n=1}^{\infty} n|d'^n_{kl}|$ and write

$$|d'^0_{kl}| + \sum_{n=1}^{\infty} n|d'^n_{kl}| \leqslant A' \tag{38}$$

where d'^n_{kl} has the same significance as d^n_{kl} of (17) with the difference that instead of k^2 and l^2 in the denominator, we have to put k^3 and l^3. Finally, instead of (14) we write

$$v'_0 = u_0\,;\ \gamma'_0 = c_0$$
$$v'_n = n^3 u_n\,;\ \gamma'_n = n^3 c_n \tag{39}$$

so that the equations (16) take the new form

$$v'_0 = \gamma'_0 - \int_0^t \sum_{k,l}^{0,\ldots,\infty} d'^0_{kl} v'_k v'_l \, ds$$

$$v'_n = \gamma'_n e^{-n^2 t} - n^3 \int_0^t e^{-n^2(t-s)} \sum_{k,l=0}^{\infty} d'^n_{kl} v'_k v'_l \, ds. \tag{40}$$

From this point onwards we have only to replace c by c' and A by A' in the above treatment and argue word for word that one and only one solution of (40) exists, and that the differential equation (6) has therefore only one solution of the form

$$u(x, t) = \sum_{n=0}^{\infty} u_n(t) \cos nx.$$

3. We shall conclude the paper after indicating how the above method can be applied to equations in several independent

variables. For simplicity consider equations in three independent variables. Let D be any bounded closed domain, simply or multiply connected, bounded by closed regular curves c in the x-y plane, and let $u(x, y, t)$ be a function of three independent variables satisfying the equation

$$\frac{\partial}{\partial x}\left(p(x, y)\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(p(x, y)\frac{\partial u}{\partial y}\right)-\frac{\partial u}{\partial t}=u^2 \tag{41}$$

for the boundary conditions

$$u(x, y, t)=0 \text{ on } c \text{ for all } t \geqslant 0 \tag{42}$$

$$u(x, y, 0)=u_0(x, y)\sim \sum_{n=1}^{\infty} c_n\omega_n(x, y) \tag{43}$$

where $p(x, y)$ is an essentially positive function, continuous and uniformly bounded together with its partial derivatives up to the second order in D and on c, and $\omega_n(x, y)$ are the eigen-functions corresponding to the eigen-values μ_n of

$$\frac{\partial}{\partial x}\left(p(x, y)\frac{\partial \omega}{\partial x}\right)+\frac{\partial}{\partial y}\left(p(x, y)\frac{\partial \omega}{\partial y}\right)+\mu\omega=0 \tag{44}$$

$$\omega(x, y)=0 \text{ on } c. \tag{45}$$

It is known* that the eigen-values μ_n are all positive and that $\Sigma\mu_n^{-2}$ is convergent. Also if we put

$$L(\omega)=\frac{\partial}{\partial x}\left(p\frac{\partial \omega}{\partial x}\right)+\frac{\partial}{\partial y}\left(p\frac{\partial \omega}{\partial y}\right)$$

and

$$a_{kl}^n=\int_D \omega_k\omega_l\omega_n dxdy \tag{46}$$

then on account of (44), Green's formula and (45)

$$\begin{aligned} a_{kl}^n &= \int_D \omega_k\omega_l\omega_n dxdy \\ &= -\frac{1}{\mu_n}\int_D \omega_k\omega_l L(\omega_n)dxdy \\ &= -\frac{1}{\mu_n}\int_D \omega_n L(\omega_k\omega_l)dxdy. \end{aligned}$$

But

$$\begin{aligned} L(\omega_k\omega_l) &= \omega_k L(\omega_l)+\omega_l L(\omega_k)+2p\left\{\frac{\partial \omega_k}{\partial x}\frac{\partial \omega_l}{\partial x}+\frac{\partial \omega_k}{\partial y}\frac{\partial \omega_l}{\partial y}\right\} \\ &= -(\mu_k+\mu_l)\omega_k\omega_l+2Q(\omega_k\omega_l), \text{ say.} \end{aligned}$$

* L. Lichtenstein *Math. Zeit.* 3 (1919) 127-160.

So

$$\frac{a^n_{kl}}{\mu_k\mu_l}=-\frac{1}{\mu_n}\Big[-\Big(\frac{1}{\mu_k}+\frac{1}{\mu_l}\Big)\int_D \omega_k\omega_l\omega_n dx dy + \frac{2}{\mu_k\mu_l}\int_D Q(\omega_k\omega_l)\omega_n dx dy\Big]$$

$$=\frac{1}{\mu_n}\Big[\Big(\frac{1}{\mu_k}+\frac{1}{\mu_l}\Big)a^n_{kl}-2b^n_{kl}\Big], \text{ say.}$$

Since $\sum_{n=1}^{\infty}\mu_n^{-2}$, $\sum_{n=1}^{\infty}(a^n_{kl})^2$ and $\sum_{n=1}^{\infty}(b^n_{kl})^2$ are convergent, it follows on account of Schwarz's inequality that the series $\sum_{n=1}^{\infty}\frac{a^n_{kl}}{\mu_k\mu_l}$ is uniformly and absolutely convergent.

Let

$$\sum_{n=1}^{\infty}\frac{|a^n_{kl}|}{\mu_k\mu_l}\leqslant A, \tag{47}$$

and let $\Sigma\mu_n c_n$ be absolutely convergent, so that we write

$$\sum_{n=1}^{\infty}\mu_n|c_n|=c. \tag{48}$$

Now if we assume a solution of (41) in the form

$$u(x, y, t)\sim\sum_{n=1}^{\infty}u_n(t)\omega_n(x, y) \tag{49}$$

then

$$u^2\sim\Sigma z_n(t)\omega_n(x, y)$$

$$z_n(t)=\sum_{k, l=1}^{\infty}a^n_{kl}u_k u_l, \tag{50}$$

and substituting (49) and (50) in (41) and equating Fourier coefficients of both sides, we have

$$\frac{du_n}{dt}+\mu_n u_n=-z_n(t).$$

$$u_n(0)=c_n. \tag{51}$$

or

$$u_n(t)=c_n e^{-\mu_n t}-\int_0^t e^{-\mu_n(t-x)}\sum a^n_{kl}u_k u_l\,dx. \tag{52}$$

Setting

$$v_n = \mu_n u_n;\ \gamma_n = c_n \mu_n \tag{53}$$

and multiplying both sides of (52) by μ_n, we have

$$v_n(t) = \gamma_n e^{-\mu_n t} - \mu_n \int_0^t e^{-\mu_n(t-s)} \sum \frac{a_{kl}^n}{\mu_k \mu_l} v_k v_l \, ds. \tag{54}$$

We can follow the method of Siddiqi and solve the equations (54) uniquely by successive approximations such that $\Sigma v_n(t)$ is uniformly and absolutely convergent for all values of t provided $4Ac < 1$.

4. We can proceed as in **2** and solve the equation (41) for the second boundary condition

$$\frac{\partial u}{\partial n}(x, y, t) = 0 \text{ on } c \text{ for all } t \geqslant 0 \tag{55}$$

$$u(x, y, 0) = u_0(x, y) = \sum_{r=0}^{\infty} c_r \omega_r(x, y) \tag{43}$$

where ω_r's are the corresponding Sturm-Liouville eigen-functions of (44) for the second boundary condition

$$\frac{\partial \omega}{\partial n} = 0 \text{ on } c, \tag{56}$$

n denoting the interior normal at a point on c. Solution is possible under the same condition as (29) where A and c have analogous significance here.

5. Clearly the above method can be used without difficulty to the equations

$$\frac{\partial}{\partial x}\left(p \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(p \frac{\partial u}{\partial y}\right) - \frac{\partial u}{\partial t} = P(x, y, t, u) \tag{57}$$

where the functional on the right is of the form

$$P(x, y, t, u) = \sum_{r=2}^{\infty} p_r(x, y, t) u^r \tag{58}$$

for the two boundary conditions. But the method seems to fail if we have on the right side of (57) a functional $P\left(x, y, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$. However a different approach to such equations where the functionals on the right are of a more general type than what is in (58) or in the right side of (1) and (2) forms the subject matter of another paper.*

* *Proc. Ind. Acad. Sci.* (*A*) 9, 479-494.

ERRATA.

In the paper on "Infinite Integrals involving Whittaker Functions" Vol. III, Pp. 169—181,

Page	Line	*For*	*Read*
170	20	are	is
171	10	$=$	$\neq$
173	7, 8	o(Gr).	0 (Zero).
,,	,,	$\pi^{\frac{1}{2}}$	$\pi^{-\frac{1}{2}}$
174	15	$M_{\frac{1}{2}\nu+\frac{1}{4}, -\frac{1}{4}}$	$M_{\frac{1}{2}\nu+\frac{1}{4}, -\frac{1}{4}}(\frac{1}{2}z^2)$
,,	20	$\Gamma(\frac{1}{2}-\mu-l-n)$	$\Gamma(\frac{1}{2}\mu-l-n)$
,,	,,	$2^{\frac{1}{2}\mu-n\frac{1}{2}}$	$2^{\frac{1}{2}\mu-n+\frac{1}{2}}$
175	16	$\frac{1}{2}+k+m+\frac{1}{2}\nu+1$	$\frac{1}{2}+k+m+\frac{1}{2}\nu$
180	16	$(\alpha+r+1)$	$\Gamma(\alpha+r+1)$
181	7	$\Gamma(\frac{1}{2}\nu\frac{1}{2})$	$\Gamma(\frac{1}{2}\nu+\frac{1}{2})$

Also, the footnotes on p. 176 should come against line 12, p. 170 and line 12, p. 174 respectively.

THE TENSOR ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS*

By D. D. KOSAMBI, *Fergusson College, Poona.*

[Received 13 May 1939]

After the comprehensive works of Bortolotti[1] on partial differential equations of the second order from the differential geometer's point of view, and the equally comprehensive memoir of Kawaguchi and Hombu[2] on systems of higher order, the present note serves only to show that slightly different results can be obtained by keeping to the point of view that I have used in my former papers[3]. The method, in particular, is to handle such systems as obey the following postulates and for the special transformation groups under which the postulates hold: (1) the system of equations transforms according to the tensor law; (2) the equations of variation of the given system are also tensorial when the variation itself is a vector; (3) there exists at least one operator which is vectorial in character and corresponds to total differentiation with respect to one of the independent variables.

To illustrate this, let us consider the second order system

$$\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + H^i_{\alpha\beta}(u,\, x,\, p^r_\nu) = 0\,;\; p^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}. \tag{1}$$

Here, the Latin indices refer to the coordinates x, and range over values $1, \ldots, n$; the Greek indices to the parameters u, and have the values $1, \ldots, m$. The functions $H^i_{\alpha\beta}$ must have the transformation law

$$-H'^i_{\alpha\beta} = -H^r_{\nu\delta} \frac{\partial x'^i}{\partial x^r} \frac{\partial u^\nu}{\partial u'^\alpha} \frac{\partial u^\delta}{\partial u'^\beta} + \frac{\partial^2 x'^i}{\partial x^r \partial x^s} p^r_\delta p^s_\nu \frac{\partial u^\delta}{\partial u'^\alpha} \frac{\partial u^\nu}{\partial u'^\beta} + \frac{\partial^2 u^\nu}{\partial u'^\alpha \partial u'^\beta} \frac{\partial x'^i}{\partial x^r} p^r_\nu. \tag{2}$$

* Read at the tenth conference of the Indian Mathematical Society, Lucknow (1938).

under the group

$$x'^i = F^i(x^1, \ldots, x^n)\ ;\ u'^\alpha = \phi^\alpha(u^1, \ldots, u^m). \tag{3}$$

But we can speak of x-transformations or u-transformations alone and of x-tensors or u-tensors accordingly. *Tensor* will mean, unless specialised, a geometric object which has the proper law of transformation for both sorts of indices. It is assumed that the conditions of integrability of the partial differential equations are identically satisfied, but no direct use will be made of them. We introduce the non-tensorial operator of differentiation with respect to u, viz.

$$\partial_\alpha \equiv \frac{\partial}{\partial u^\alpha} + p^i_\alpha \frac{\partial}{\partial x^i} - H^i_{\alpha\beta} \frac{\partial}{\partial p^i_\beta}. \tag{4}$$

It follows that for an x-vector λ^i, a vectorial operator must be of the type $D_\alpha\lambda^i \equiv \partial_\alpha\lambda^i + \gamma^i_{\alpha r}\lambda^r$. But this will not do for any tensor with Greek indices. Therefore, the $\gamma^i_{\alpha j}$ must behave like covariant u-vectors, and an additional term will have to enter the D_α. We may, therefore, take the general operator to be of the form

$$\left.\begin{aligned} D_\alpha T^{\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot} = \partial_\alpha T^{\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot} + \gamma^i_{\alpha r} T^{r\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot} - \gamma^r_{\alpha j} T^{\cdot\,\cdot\,\cdot}_{r\,\cdot\,\cdot} \\ + \Gamma^\nu_{\alpha\rho} T^{\cdot\,\cdot\,\rho}_{\cdot\,\cdot\,\cdot} - \Gamma^\rho_{\alpha\sigma} T^{\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\rho} \end{aligned}\right\} \tag{5}$$

The laws of transformation for the two sets of coefficients must be as follows:

$$\begin{aligned} \gamma^s_{\sigma r}\frac{\partial x'^i}{\partial x^s}\frac{\partial u^\sigma}{\partial u'^\alpha} &= \gamma'^i_{\alpha s}\frac{\partial x'^s}{\partial x^r} + p^s_\sigma \frac{\partial^2 x'^i}{\partial x^r \partial x^s}\frac{\partial u^\sigma}{\partial u'^\alpha}. \\ \Gamma'^\sigma_{\alpha\beta}\frac{\partial u^\nu}{\partial u'^\sigma} &= \Gamma^\nu_{\sigma\tau}\frac{\partial u^\sigma}{\partial u'^\alpha}\frac{\partial u^\tau}{\partial u'^\beta} + \frac{\partial^2 u^\nu}{\partial u'^\alpha \partial u'^\beta}. \end{aligned} \tag{6}$$

Let the coordinates x^i now undergo a vector variation represented by $\bar{x}^i = x^i + \varepsilon\lambda^i$. Neglecting the coefficients of ε^2 and higher powers as usual, we have the equations of variation

$$\partial_\alpha\partial_\beta\lambda^i + \partial_\nu\lambda^r \frac{\partial H^i_{\alpha\beta}}{\partial p^r_\nu} + \lambda^r \frac{\partial H^i_{\alpha\beta}}{\partial x^r} = 0. \tag{7}$$

These can at once be put in the invariantive form

$$D_\alpha D_\beta \lambda^i + D_\nu\lambda^r T^{i\nu}_{r\alpha\beta} + \lambda^r P^i_{r\alpha\beta} = 0, \tag{8}$$

where

$$T^{i\nu}_{j\alpha\beta} = \delta^i_j \Gamma^\nu_{\alpha\beta} - \gamma^i_{\alpha j}\delta^\nu_\beta - \gamma^i_{\beta j}\delta^\nu_\alpha + \frac{\partial H^i_{\alpha\beta}}{\partial p^j_\nu}\ ; \tag{8-a}$$

$$P^i_{j\alpha\beta} = \frac{\partial H^i_{\alpha\beta}}{\partial x^j} - \frac{\partial H^i_{\alpha\beta}}{\partial p^r_\sigma}\gamma^r_{\sigma j} + \gamma^r_{\beta j}\gamma^i_{\alpha r} - \partial_\beta \gamma^i_{\alpha j}. \tag{8-b}$$

In my former work, the coefficients $\gamma^i_{\alpha j}$ were determined by taking the tensor corresponding to the first of these as equal to zero. This can no longer be done here, as it would be too restrictive, and would not even then serve to determine both $\gamma^i_{\alpha j}$ and $\Gamma^\nu_{\alpha\beta}$. $\Gamma^\nu_{\alpha\beta}$ need not be symmetrical in the subscripts, but the anti-symmetrical part (torsion) will be indeterminate; one may therefore assume $\Gamma^\nu_{\alpha\beta}$ to be symmetric. It is also clear that the derivatives $\dfrac{\partial^2\gamma^i_{\alpha j}}{\partial p^k_\beta \partial p^l_\nu}$ and $\dfrac{\partial \Gamma^\nu_{\alpha\beta}}{\partial p^i_\sigma}$ are the components of tensors. The operator $\dfrac{\partial}{\partial p^i_\alpha}$ is also tensorial in character, adding a covariant x-component and a contravariant u-component.

The problem is now to determine the coefficients $\gamma^i_{\alpha j}$ and $\Gamma^\nu_{\alpha\beta}$ in a fashion which has some claim to be called intrinsic. Let us assume that all possible contractions of $T^{i\nu}_{j\alpha\beta}$ vanish. This gives

$$T^{r\nu}_{r\alpha\beta} \equiv n\Gamma^\nu_{\alpha\beta} - \delta^\nu_\beta \gamma^r_{\alpha r} - \delta^\nu_\alpha \gamma^r_{\beta r} + \frac{\partial H^r_{\alpha\beta}}{\partial p^r_\nu} = 0$$

$$T^{r\nu}_{r\alpha\nu} \equiv n\Gamma^\nu_{\alpha\nu} - (m+1)\gamma^r_{\alpha r} + \frac{\partial H^r_{\alpha\nu}}{\partial p^r_\nu} = 0, \text{ etc.} \tag{9}$$

The general solutions must then have the form;

$$\gamma^i_{\alpha j} = \frac{1}{m+1}\left[\Gamma^\sigma_{\alpha\sigma}\delta^i_j + \frac{\partial H^i_{\alpha\sigma}}{\partial p^j_\sigma}\right];$$

$$\Gamma^\nu_{\alpha\beta} = -\frac{1}{n}\frac{\partial H^r_{\alpha\beta}}{\partial p^r_\nu} + \delta^\nu_\beta \tau_\alpha + \delta^\nu_\alpha \tau_\beta. \tag{10}$$

The τ_α that enter into the second of these must have the law of transformation

$$\tau'_\alpha = \tau_\nu \frac{\partial u^\nu}{\partial u'^\alpha} - \frac{1}{n}\frac{\partial^2 x'^j}{\partial x^r \partial x^s}\frac{\partial x^r}{\partial x'^j} p^s_\nu \frac{\partial u^\nu}{\partial u'^\alpha}, \tag{11}$$

but are otherwise arbitrary so far as the present argument is concerned. This shows, in the first place, that *the logical types of connection for our systems are not affine but projective*; however we shall not pursue this further.

There exists another set of the equations of variation, namely those obtained by giving a vector variation to u. These are

$$p^i_\nu D_\alpha D_\beta \mu^\nu + D_\nu \mu^\sigma Q^{i\nu}_{\alpha\beta\sigma} + \mu^\sigma R^i_{\alpha\beta\sigma} = 0, \tag{12}$$

if $u^\alpha = \bar{u}^\alpha + \eta\mu^\alpha$, η being an infinitesimal,
where

$$Q^{i\nu}_{\alpha\beta\sigma} = \frac{\partial H^i_{\alpha\beta}}{\partial p^j_\nu} p^j_\sigma - H^i_{\alpha\sigma}\delta^\nu_\beta - H^i_{\beta\sigma}\delta^r_\alpha + \Gamma^\nu_{\alpha\beta} p^i_\sigma - p^i_\rho \Gamma^\rho_{\alpha\sigma}\delta^\nu_\beta - p^i_\rho \Gamma^\rho_{\beta\sigma}\delta^\nu_\alpha$$

$$R^i_{\alpha\beta\sigma} = p^i_\nu(\Gamma^\nu_{\delta\sigma}\Gamma^\delta_{\alpha\beta} - \Gamma^\nu_{\alpha\delta}\Gamma^\delta_{\beta\sigma} - \partial_\beta \Gamma^\nu_{\alpha\sigma}) - \frac{\partial H^i_{\alpha\beta}}{\partial u^\sigma} - \Gamma^\delta_{\nu\sigma} Q^{i\nu}_{\alpha\beta\delta}. \qquad (13)$$

The first of these might also be used to determine the connection $\Gamma^\nu_{\alpha\beta}$. But it is clear, from the manner in which p^i_ν enters, that this cannot be done without differentiation. In particular, inasmuch as $\frac{\partial \Gamma^\nu_{\alpha\beta}}{\partial p^i_\delta}$ is a tensor, we might set $\frac{\partial Q^{i\sigma}_{\alpha\beta\sigma}}{\partial p^i_\nu} + p^i_\sigma \frac{\partial \Gamma^\sigma_{\alpha\beta}}{\partial p^i_\nu} = 0$, which leads at once to

$$\Gamma^\nu_{\alpha\beta} = \frac{1}{n}\left[\frac{\partial^2 H^i_{\alpha\beta}}{\partial p^j_\sigma \partial p^i_\nu} p^j_\sigma - \frac{\partial H^i_{\alpha\beta}}{\partial p^i_\nu}\right]. \qquad (14)$$

This connection involves the *second* partial derivatives of $H^i_{\alpha\beta}$. With the first derivatives alone, we cannot go beyond (10). Since the difference of two sets of Γ's is a tensor, the differential invariants of the space may be calculated for any one connection, and are then obtained for any other by the use of this tensor-difference.

There is another operator besides D_ν and $\partial/\partial p^i_\alpha$, but it may be obtained by alternating the pair thus:

$$\left[\frac{\partial}{\partial p^j_\nu} D_\nu T^{i\alpha} - D_\nu \frac{\partial}{\partial p^j_\nu} T^{i\alpha}\right] = m\frac{\partial T^{i\alpha}}{\partial x^j} + \frac{\partial T^{i\alpha}}{\partial p^r_\beta}\left[\gamma^r_{\beta j} - \Gamma^\nu_{\nu\beta}\delta^r_j - \frac{\partial H^r_{\nu\beta}}{\partial p^j_\nu}\right]$$

$$+ T^{r\alpha}\frac{\partial \gamma^i_{\nu r}}{\partial p^j_\nu} + T^{i\beta}\frac{\partial \Gamma^\alpha_{\nu\beta}}{\partial p^j_\nu}. \qquad (15)$$

Discarding the additive tensorial terms and making use of (10), we get the simplified operator

$$\underset{j}{\nabla} T^{i\alpha} = \frac{\partial T^{i\alpha}}{\partial x^j} - \gamma^r_{\beta j}\frac{\partial T^{i\alpha}}{\partial p^r_\beta} + \frac{1}{m} T^{r\alpha}\frac{\partial \gamma^i_{\nu r}}{\partial p^j_\nu}. \qquad (16)$$

The differential invariants of the space are to be obtained by alternating the three operators given, as usual.

The usefulness of the foregoing discussion lies in its adaptability to the case of differential equations of higher order. Let, for instance, such a system be given by

$$\frac{\partial^{q+1} x^i}{\partial u^{\alpha_1}\partial u^{\alpha_2}\ldots\partial u^{\alpha_{q+1}}} + H^i_{\alpha_1\alpha_2\ldots\alpha_{q+1}}(u, x, p^i_\alpha, \ldots, p^i_{\alpha_1\ldots\alpha_q}) = 0. \quad (17)$$

The operator D_ν will be of the same type as before. The connection coefficients can again be determined from the two sets of equations of variation, in particular, by contraction of the corresponding coefficients of the varied equations. The remaining coefficients of the equations of variation give the "primary" differential invariants of the system. A new differential operator is obtained by alternating and contracting $\partial/\partial p^i_{\alpha_1 \cdots \alpha_q}$ and D_ν. This will give an operator with one covariant Latin index and $q-1$ contravariant Greek indices, viz. $\nabla_i^{\alpha_1 \cdots \alpha_{q-1}}$. Alternation and contraction of this $\nabla_i^{\alpha_1 \cdots \alpha_{q-1}}$ with D_ν will again get rid of another Greek index, and give a second operator $\nabla_i^{\alpha_1 \cdots \alpha_{q-2}}$. This can be continued till no Greek indices are left, and we obtain the operator which corresponds to the purely Latin index covariant derivative ∇_i. Further alternations will give only differential invariants. At each stage, additive tensorial terms can be discarded to obtain a reduced operator.

The complete set of differential invariants is not worked out here in view of the memoirs already cited. But it must consist, for the greater part, of those that enter into the equations of variation p^i_α, and such others as are to be obtained from these by the application of the tensorial operators.

REFERENCES.

(1) Enea Bortolotti: *Rendiconti della Reale Accademia Nazionale dei Lincei*, 23 (1936), 16-21; 104-10; 175 80.

(2) A. Kawaguchi and H. Hombu: *Journal of the Faculty of Science, Hokkaido Imperial University*, 1 (1937), 6, 21-62.

(3) D. D. Kosambi: *Quarterly Journal of Mathematics*, (Oxford), 6 (1935), 1-12 and 7 (1936), 97-104.

THE THEORY OF THE GENERAL CONTACT CIRCLES OF A TRIANGLE

By K. RANGASWAMI*, *Annamalai University.*

[Received 17 April 1939]

Introduction. With every point P in the plane of a triangle may be associated two circles—its *pedal* and *contact*† circles. The former is the circle through the projections of P on the sides of the triangle, while the latter is the circle through the points of contact with the sides, of the unique inconic of the triangle whose centre is P. It is known that a pair of isogonal conjugates P, P' w. r. t. a triangle determine a common pedal circle and a common contact circle. Now, pairs of isogonal conjugates w. r. t. a triangle are but the singular members of a linear ∞^3-system of conic envelopes—called *infocal*‡ *conics* (3.1)—inpolar to the rectangular hyperbolas through the in-and ex-centres of the triangle. This suggests the possibility of associating with every infocal conic a pedal circle and a contact circle.

The pedal circle of an infocal conic Σ may be defined as the mutual orthoptic circle of Σ and its confocal inscribed in the triangle. This definition has enabled the present writer § to derive the known properties of pedal circles as particular cases of those of pedal circles of infocal conics. In this paper *we define the contact circle of an infocal conic* Σ *as the harmonically conjugate conic of the reciprocal of* Σ *w. r. t. the* (*Steiner*) *maximum inscribed ellipse of the triangle* (3.3), and establish the properties of contact circles analogous to those of the pedal circles.

The results on contact circles obtained in this paper are given below along with the corresponding known results on pedal circles:

* I wish to acknowledge my indebtedness to Prof. A. Narasinga Rao for guidance in preparing this paper.

† The term *contact circle* was introduced in the geometry of the triangle by M. Bhimasena Rao who also proved that a pair of isogonal conjugates have the same contact circle. Vide *J. I. M. S.* (1) 1, 244.

‡ The term *infocal* is due to J. P. Gabbat. Vide *Proc. Camb. Phil. Soc.* 22 p. 752.

§ K. Rangaswami, A General Extension of Feuerbach's Theorem. *Math. Student* 6(1938), 93-8.

The contact (pedal) circles of infocal conics touching a line* Λ *cut orthogonally a fixed circle—called the orthocontact circle (orthopolar circle) of* Λ. (4.1)

The contact (pedal)† circles of infocal conics inscribed in an isogonal quadrilateral belong to a coaxal system; and the orthocontact (orthopolar) circles of the sides of the quadrilateral belong to the conjugate coaxal system. (4.2)

The contact (pedal)‡ circles of infocal conics passing through the symmedian point (circum-centre) touch the nine-points circle. (5.2)

The contact (pedal) circle of an inconic σ *of the triangle is a line called the contact line of* σ *(is the director circle of* σ*), i.e. a circle cutting orthogonally the point circle at infinity (the polar circle of the triangle).* (3.4)

1. *Harmonically conjugate conics.*

Let

$$C: \quad c_{11}\xi_1^2 + \ldots + 2c_{23}\xi_2\xi_3 + \ldots = 0$$

be the equation of a conic either in point or line coordinates ξ_1, ξ_2, ξ_3 referred to a triangle ABC, say Δ. We call the conic

$$\bar{C}: \quad c_{11}\xi_1^2 + \ldots - 2c_{23}\xi_2\xi_3 - \ldots = 0$$

the *harmonically conjugate*§ (*h.c.*) conic of C w. r. t. Δ. In particular, when the conic locus is a squared line Λ, the *h.c.* conic is an inconic σ of Δ which will be termed the *pole conic* of Λ w. r. t. Δ; while, if the conic envelope is a squared point P, the *h.c.* conic is the *polar conic* of P w. r. t. Δ.

It follows from the definition of *h.c.* conics that if a conic locus S is outpolar to a conic envelope Σ, then $\bar{S}$ is outpolar to $\bar{\Sigma}$. Now, if P be any point on S, S is outpolar to the squared point P. Therefore, $\bar{S}$ is outpolar to the polar conic of P. Similarly if Λ be any tangent to Σ, $\bar{\Sigma}$ is inpolar to the pole conic of Λ. We thus have the following

Theorem. *If two conic loci (envelopes) be harmonically conjugate w. r. t. a triangle* Δ, *then either of them is outpolar*

* V. Ramaswami Aiyar, The Orthopolar Theory, *J. I. M. S.* (1) 4, 214.

† *Math. Student. loc. cit.* p. 97.

‡ *Math. Student. loc. cit.* p. 96.

§ The term *harmonically conjugate conic* is due to Prof. Baker. Vide *Principles of Geometry*, Vol. 2, p. 122.

(inpolar) to the polar conics (pole conics) w. r. t. Δ of points on (tangents to) the other. (1.1)

Let a conic S meet the sides of Δ in x, x_1; y, y_1; z, z_1 and let $\bar{x}, \bar{x}_1$ be the respective harmonic conjugates of x, x_1 w. r. t. B, C; $\bar{y}, \bar{y}_1$ of y, y_1 w. r. t. C, A; $\bar{z}, \bar{z}_1$ of z, z_1 w. r. t. A, B. By (1.1), since x lies on S, $\bar{S}$ is outpolar to the polar conic of x, i.e. to the line pair BC, $A\bar{x}$; hence $\bar{S}$ passes through $\bar{x}$. Thus, the six points $\bar{x}, \bar{x}_1$, etc. all lie on $\bar{S}$. In the particular case when S touches the sides of Δ at x, y, z the lines Ax, By, Cz are concurrent at a point T and from the above we see that the *h.c.* conic of S is the polar line of T w. r. t. Δ (as also w. r. t. S) repeated twice.

2. *Preliminary theorems on polar and pole conics of a triangle.*

Let Λ be any line, Q its pole w. r. t. Δ, σ the pole conic of Λ and xyz the contact triangle of σ with the sides of Δ. Now if P be any point on Λ, its polar conic π passes through A, B, C, Q and hence has xyz for a self-polar triangle. Therefore, π is inpolar to σ. Again, the triangle ABC is inscribed in π and circumscribed to σ. Hence π should also be outpolar to σ. Thus

The polar conics of points on a line Λ are doubly apolar to the pole conic of Λ w. r. t. Δ. (2.1)

Suppose that P, P' are two points on Λ and π, π' their polar conics. Let QR, QR' be the tangents to π, π' at Q. Then, since Q is the pole of Λ w. r. t. σ it follows from (2.1) that QR is the polar of P w. r. t. σ. Similarly QR' is the polar of P' w. r. t. σ. Further since both the conics π, π' have QR, QR' for a pair of conjugate lines and xyz for a self-conjugate triangle, any conic in the tangential system determined by them will also have these properties. By reciprocating w. r. t. σ we have the result:

If P, P' be two points in the plane of a triangle Δ and σ the pole conic of PP', the range determined by the polar conics of P, P' w. r. t. Δ is the reciprocal w. r. t. σ of the pencil determined by conics having Δ for a self-polar triangle and outpolar to the point-pair P, P' (2.2)

Let Λ, Λ′ be any two lines and σ, σ′ their pole conics. It is easy to see that the pole* of Λ w. r. t. σ′ is also the pole of Λ′ w. r. t. σ. Now, if Λ′ is a variable line through a fixed point P, the

* This point may be called the mixed pole of Λ, Λ′ w. r. t. Δ.

poles of Λ w. r. t. the pole conics σ' lie on the polar of P w. r. t. σ. Further by (2.1), the conics σ' are also inpolar to the polar conic of P. We have thus the result:

The pole conics of lines through a point P belong to a range inpolar to the polar conic of P; and the poles of any line Λ w. r. t. the conics of the range lie on the polar of P w. r. t. the pole conic of Λ. (2.3)

3. *The general contact circle.*

Consider a metrical plane, the absolute Ω being the pair of circular points i, j of the plane. The conics in the plane having a given triangle Δ for a self-polar triangle and outpolar to Ω belong to a pencil and will be termed the *principal hyperbolas* of Δ. Now, if σ is an inconic of Δ, it is inpolar to the principal hyperbolas; so will be any conic Σ confocal with σ. Since Σ is a confocal to an inconic, we say that Σ is an *infocal* to Δ. Thus

The infocal conics of a triangle are the linear ∞^3-system of conic envelopes inpolar to the principal hyperbolas. (3.1)

Now, the pole conic of the line at infinity is the conic, say Γ, touching the sides of the triangle at their mid-points A', B', C', namely, the maximum inscribed ellipse of the triangle. Since i, j lie on the line at infinity, we have by (2.2) the following

THEOREM. *The principal hyperbolas are the reciprocals w. r. t. the maximum inscribed ellipse of the conics of the range determined by the polar conics of the circular points w. r. t. the triangle.* (3.2)

Since any circle S in the plane is a conic through i, j it follows from (1.1) that $\bar{S}$ is outpolar to the polar conics of i, j. Hence by (3.2), the reciprocal conic Σ of $\bar{S}$ w. r. t. Γ is inpolar to the principal hyperbolas, i.e. Σ is an infocal conic of Δ. We thus have our main

THEOREM. *The harmonically conjugate conic of the reciprocal of an infocal conic Σ w. r. t. the maximum inscribed ellipse is a circle S. We call the circle S the contact circle of the infocal conic Σ w. r. t. the triangle.* (3.3)

When the infocal is an inconic σ, its reciprocal $\bar{S}$ w. r. t. Γ is a circumconic of the medial triangle $A'B'C'$. Hence the *h.c.* conic of $\bar{S}$ in this case consists of a line Λ and the line at infinity. We thus have the

THEOREM. *The contact circle of an inconic σ consists of a line Λ and the line at infinity. We call Λ the contact line of the inconic σ w. r. t. the triangle.* (3.4)

Let Λ be any line and σ its pole conic. The pole Q of Λ w. r. t. Γ is the centre of σ. If xyz is the contact triangle of σ, the *h.c.* conic of the circle xyz consists of the line Λ and another line Λ'. Hence, the reciprocal of the conic locus consisting of the two lines Λ, Λ' is by (3.3) the conic-envelope consisting of the point Q and its isogonal conjugate Q' w. r. t. Δ. Thus the circle xyz is the contact circle of the isogonal conjugates Q, Q'. We are thus led to the following definition*, viz.

The contact circle of a pair of isogonal conjugates Q, Q' w. r. t. a triangle is the circle through the points of contact of the inconics whose centres are Q, Q' with the sides of the triangle. (3.5)

We next proceed to specify the infocal conics whose contact circles are point circles. If S is a point circle it consists of two circular lines, one through each circular point. Now, it is easily seen that the condition that S may break up into a pair of lines (either into two circular lines or into an arbitary line and the line at infinity) is that the *h.c.* conic of Σ w. r. t. the medial triangle $A'B'C'$ should be a point-pair. But if S consists of an arbitrary line and the line at infinity, the corresponding infocal is by (3.4) an inconic of Δ. We thus obtain that

The necessary and sufficient condition that the contact circle of an infocal conic Σ† may be a point circle is that the h.c. conic of Σ w. r. t. the medial triangle should break up into a pair of points. (3.6)

It may be shown from (3.6) that the point circles which are the contact circles of pairs of isogonal conjugates are the points on the circles on BC, CA, AB as diameters.

4. *The Orthocontact circle.*

Consider the infocals Σ that touch a line Λ. Their reciprocal conics $\bar{S}$ w. r. t. Γ will pass through the pole P of Λ w. r. t. Γ. Hence by (1.1) the *h.c.* conics of $\bar{S}$, namely, the contact circles of the infocals Σ will be outpolar to the polar conic π of P, and

* We notice that this is the definition of the contact circle of isogonal conjugates first given by Bhimasena Rao.

† We suppose that the infocal Σ is not an inconic of Δ.

therefore by Gaskin's theorem will cut orthogonally the director circle of π. We have thus the

THEOREM*. *If Λ be any line and P its pole w. r. t. the maximum inscribed ellipse, the contact circles of infocal conics touching Λ cut orthogonally the director circle of the polar conic of P w. r. t. the triangle. This circle will be termed the orthocontact circle of the line Λ.* (4.1)

Let Σ, Σ' be any two infocal conics; then any member Σ'' in the range determined by them is also an infocal conic (3.1). Let a, b, c, d be the common tangents of Σ, Σ' and α, β, γ, δ their orthocontact circles. By (4.1) α, β, γ, δ cut orthogonally the contact circles, say, S, S' of Σ, Σ' and hence belong to a coaxal system. If Σ'' be any infocal in the range determined by Σ, Σ', its contact circle S'' cuts orthogonally the circles α, β, γ, δ. Thus, the circles S, S', S'' belong to the coaxal system conjugate to the system of orthocontact circles. Again, the pairs of opposite vertices of the quadrilateral formed by the lines a, b, c, d are isogonal conjugates w. r. t. Δ. If we call a quadrilateral whose three pairs of opposite vertices are isogonal conjugates *an isogonal quadrilateral* of the triangle, we have the

THEOREM. *The contact circles of infocal conics inscribed in an isogonal quadrilateral of a triangle belong to a coaxal system and the orthocontact circles of the sides of the quadrilateral belong to the conjugate coaxal system.* (4.2)

Now, in the range determined by two infocal conics Σ, Σ' there is a unique conic σ inscribed in the triangle, and its contact circle by (3.4) is a line Λ. By (4.2), Λ must belong to the coaxal system determined by the contact circles of Σ, Σ'. Thus Λ is the radical axis of this coaxal system. When one of the infocal conics, say Σ', is the absolute conic Ω, we have by (4.2) the result:

The contact circles of conics confocal with an inconic σ belong to a coaxal system whose radical axis is the contact line of σ. (4.3)

5. *Extension of Feuerbach's theorem.*

The *h.c.* conic of the nine-points circle of the triangle consists of the radical axis of the circumcircle and the nine-points circle, namely, the Lemoine line of the triangle and the line at infinity.

* This is the analogue of Ramaswamy Aiyar's extension of Lemoine's theorem. Vide *J. I. M. S.* (1) 4, 214.

By (1.1) the nine-points circle must be outpolar to the polar conics of points on the Lemoine line. Since the pole w. r. t. Δ of the Lemoine line is the orthocentre, these conics are the rectangular hyperbolas through ABC and their centres lie on the nine-points circle.

Let a circle S cut the nine-points circle in P, Q and let $\bar{S}$ cut the Lemoine line in P', Q'. Then, the polar conic of P' w. r. t. Δ is a rectangular hyperbola through ABC whose centre must be one of the points P, Q, say P. Similarly, the centre of the polar conic of Q' is Q. If, however, P', Q' coincide P, Q will also coincide. We have thus the following

THEOREM. *If a circle touches the nine-points circle of a triangle, its h.c. conic touches the Lemoine line of the triangle.* (5.1)

The pole conic of the Lemoine line touches the sides of Δ at the feet of the altitudes, and its centre K is the symmedian point of the triangle. But K is also the pole of the Lemoine line w. r. t. Γ. Now, if an infocal Σ passes through K its reciprocal $\bar{S}$ w. r. t. Γ touches the Lemoine line; therefore by (5.1) the contact circle of Σ touches the nine-points circle. We are thus led to the following result which constitutes an extension of Feuerbach's theorem:

THEOREM. *If an infocal conic passes through the symmedian point of a triangle, its contact circle touches the nine points circle of the triangle.* (5.2)

As a particular case when the infocal conic is a pair of isogonal conjugates, we infer from (5.2) the result: *If a pair of isogonal conjugates are collinear with the symmedian point their contact circle touches the nine-points circle.* (5.21)

Consider the infocal conics that pass through a point T. Their reciprocals w. r. t. Γ touch the polar line Λ of T w. r. t. Γ. Let Q be the pole of Λ w. r. t. Δ, σ the pole conic of Λ and xyz the contact triangle of σ. Now, the centre of σ is the pole of Λ w. r. t. Γ, i.e. the point T. Thus the circle xyz is the contact circle of T w. r. t. Δ. Also the polar conics of points on Λ pass through A, B, C, Q and the locus of their centres is the conic ϕ through x, y, z, A', B', C'.

Consider a circle S, and let $\bar{S}$ cut Λ in P, P'. By (1.1) the polar conics, say π, π' of P, P' are inpolar to S and therefore their director circles cut orthogonally the circle S. Further, since

xyz is a self-polar triangle for π, π' the director circles of π, π' also cut orthogonally the circle xyz. Thus the centres of π, π' are the intersections of ϕ with the radical axis of the circle S and the circle xyz. In particular, when $\bar{S}$ touches Λ, S is the contact circle of an infocal passing through T and in this case the radical axis of the circle S and the circle xyz touches ϕ. We thus conclude:

If T be any point in the plane of a triangle ABC, Q the pole w. r. t. Δ of the polar of T w. r. t. the maximum inscribed ellipse and ϕ the centre locus of conics through $ABCQ$, the radical axis of the contact circles of the point T and any infocal through T touches the conic ϕ. (5.3)

When T is the symmedian point of the triangle, the contact circle of T as well as the centre-locus ϕ is the nine-points circle, and we have therefore the result contained in (5.2). When the point T is the centroid of the triangle, the contact circle of T is the nine-points circle while the conic ϕ becomes the conic Γ. By (5.3) we infer:

If an infocal conic Σ passes through the centroid of the triangle, the radical axis of the nine-points circle and the contact circle of Σ touches the maximum inscribed ellipse of the triangle.

ON A PARTICULAR REPRESENTATION OF INTEGERS AS SUMS OF kth POWERS

By K. SAMBASIVA RAO, *Andhra University.*

[Received 6 March 1939]

1. *Introduction.* Consider the following way of representation of any positive integer x in the form

$$x = x_1^k + x_2^k + \ldots + x_s^k, \tag{1}$$

where $x_1, x_2, \ldots, x_s$ are integers given by

$$\begin{aligned} (x_1+1)^k &> x \geqslant x_1^k \\ (x_2+1)^k &> x - x_1^k \geqslant x_2^k \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{2}$$

the process terminating with

$$(x_s+1)^k > x - x_1^k - x_2^k - \ldots - x_{s-1}^k = x_s^k.$$

It can easily be seen that there is one and only one way of representation of an integer x in this manner. The number of kth powers required in the representation of x in the above manner is clearly a function of x and k, and hence can be denoted by $S_k(x)$.

In what follows $c_1, c_2, \ldots$, always denote positive constants independent of x but may depend on k, and $x, x_1, x_2, \ldots$ always denote positive integers.

The purpose of the present paper is to prove the following theorems:—

THEOREM* 1. *For all* $x > x_0$

$$S_k(x) < c_1 \log\log x.$$

THEOREM 2. *There exist infinitely many x for which*

$$S_k(x) > c_2 \log\log x.$$

2. *Proof of Theorem* 1. From (2) we have

$$x_1^k \leqslant x < (x_1+1)^k.$$

* I am indebted to Dr. Chowla for the proof of this theorem for the case $k=2$.

Therefore

$$\begin{aligned} 0 \leqslant x - x_1^k &< (x_1+1)^k - x_1^k \\ &= kx_1^{k-1} + \ldots + 1 \\ &< x_1^{k-1}(k + k^c + \ldots + 1) \\ &< 2^k x_1^{k-1} \\ &\leqslant 2^k x^{(k-1)/k} \end{aligned} \tag{3}$$

Similarly we have.

$$\begin{aligned} 0 \leqslant x - x_1^k - x_2^k &< 2^k (2^k x^{(k-1)/k})^{(k-1)/k} \\ &= 2^{k+k-1} x^{\{(k-1)/k\}^2} \end{aligned}$$

and

$$\begin{aligned} 0 \leqslant x - x_1^k - x_2^k - x_3^k &< 2^k \left\{ 2^{k+k-1} x^{(k-1l/k)^2} \right\}^{k-1/k} \\ &< 2^{k+k-1+(k-1)^2/k} x^{(k-1/k)^3}. \end{aligned}$$

Proceeding in this manner, we have

$$\begin{aligned} 0 &< x - x_1^k - x_2^k - \ldots - x_r^k \\ &< 2^{k+k-1+(k-1)^2/k+\ldots+(k-1)^{r-1}/k^{r-2}} x^{(k-1/k)^2} \\ &< 2^{k^2} x^{(k-1/k)^r}. \end{aligned} \tag{4}$$

Now there are two possibilities:—

Either (1) there is no x for which an $r \leqslant s$ can be found such that

$$r > \frac{\log\log x - \log\log\,(1+1/2^{k^2})}{\log\,[k/(k-1)]} \tag{5}$$

or (2) there is sometimes an x for which there is an $r \leqslant s$ with the property (5). In the first case the theorem follows at once. In the second case for an r with the property (5)

$$2^{k^2} x^{(1-1/k)^r} < 2^{k^2} + 1.$$

Hence from

$$\begin{aligned} x_{r+1}^k + \ldots + x_s^k &= x - x_1^k - \ldots - x_r^k \\ &< 2^{k^2} + 1 \end{aligned}$$

we get

$$s - r < 2^{k^2} + 1$$

Hence

$$s_k(x) < r+2^{k^2}+1$$

$$< \frac{\log\log x - \log\log(1+1/2^{k^2})}{\log(k/k-1)} + 2^{k^2}+1$$

$$< c_1 \log\log x$$

for all sufficiently large x.

3. *Proof of Theorem* 2. If we can choose a sequence of positive integers

$$x_1, x_2, \ldots, x_s \tag{6}$$

such that

$$\begin{aligned} x_1^k &< kx_2 \\ x_1^k+x_2^k &< kx_3 \\ &\cdots\cdots\cdots \\ x_1^k+x_2^k+\ldots+x_{s-1}^k &< kx_s \end{aligned} \tag{7}$$

and put

$$x_s^k+x_{s-1}^k+\ldots+x_1^k = x, \tag{8}$$

then x in (8) is represented as specified in (1) and (2). For from (7) and (8) follow

$$\begin{aligned} x_s^k &< x = x_s^k+\ldots+x_1^k < x_s^k+kx_s < (x_s+1)^k \\ x_{s-1}^k &< x-x_s^k = x_{s-1}^k+\ldots+x_1^k < x_{s-1}^k+kx_{s-1} < (x_{s-1}+1)^k \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{9}$$

Now we choose the sequence of integers (6) satisfying the conditions (7) in the following manner:—

Let x_1 be any integer $> k$. (10)

We choose x_2 such that

$$x_2 > x_1^k/k.$$

This is satisfied if we take

$$x_2 = x_1^k. \tag{11}$$

Next x_3 is chosen according to (7) such that

$$x_3 > \frac{x_1^k+x_2^k}{k}.$$

But by (10)

$$\frac{x_1^k+x_2^k}{k} < \frac{kx_2+x_2^k}{k} < x_2^k.$$

Hence x_3 can be chosen such that

$$x_3 = x_2^k = x_1^{k^2}.$$

Proceeding in this manner we choose

$$x_r = x_1^{k^{r-1}}$$

and lastly

$$x_s = x_1^{k^{s-1}}. \tag{12}$$

Hence the series of numbers

$$x_1, x_2, \ldots, x_s$$

thus chosen is that given by (6) satisfying the conditions (7). Here it is important to note that there are infinitely many values possible for both x_1 and s, and hence for

$$x = x_s^k + x_{s-1}^k + \ldots + x_1^k. \tag{13}$$

Now Theorem 2 will be completely proved by showing that for all x given by (13), the theorem is satisfied. From (12) follows

$$s = \frac{\log\log x_s - \log\log x_1 + \log k}{\log k}. \tag{14}$$

But from the first part of (9), we have

$$x_s > x^{1/k} - 1 > c_3 x^{1/k}. \tag{15}$$

Hence from (14) and (15), we have

$$s > c_2 \log\log x.$$

A NOTE ON THE PAPER OF SAMBASIVA RAO

By S. S. PILLAI.

[Received 18 April 1939]

1. In the notation of the previous paper*, Mr. Rao's results are sharpened to the following

THEOREM.

$$\text{Max } S_k(x) = \frac{\log\log x}{\log\{k/(k-1)\}} + M(k) + O(k \log k),$$

where $M(k) = \text{Max } S_k(x)$ *when* $1 \leqslant x \leqslant (2k)^k$; *and the constant in O is independent of k and x.*

2. Let K be the number $\leqslant (2k)^k - 1$, for which $S_k(K)$ is the maximum. Then

$$S_k(K) = M(k). \tag{1}$$

Further, let

$$v = 1/(k-1)\,;\; t = k^v;\; m = 1/t;\; A_r = x_1^k + x_2^k + \ldots + x_r^k + K,$$

$$x_1 = 2k\,;\; x_{r+1} = [(A_r/k)^v] + 1\,;\; \text{and } x = A_s. \tag{2}$$

Now

$$A_r \leqslant kx_{r+1}^{k-1}.$$

So

$$x_r^k \leqslant A_r = x_r^k + A_{r-1} \leqslant x_r^k + kx_r^{k-1} < (x_r+1)^k.$$

Therefore, x_r^k is the highest kth power $\leqslant A_r$.

Further $x_1^k > K$ and so

$$S_k(x) = s + M(k). \tag{3}$$

If

$$A \geqslant (2k)^k,\; (k^{-v} + A^{-v})^k + A^{-v} \leqslant k^{-v} + 2A^{-v} \leqslant 1. \tag{4}$$

Hence, since $A_r \geqslant (2k)^k$,

$$A_{r+1} = x_{r+1}^k + A_r \leqslant \{(A_r/k)^v\}^k + A_r$$

$$= A_r^t\{(k^{-v} + A_r^{-v})^k + A_r^{-v}\} \leqslant A_r^t,$$

so

$$x = A_s \leqslant A_{s-1}^t \leqslant A_{s-2}^{t^2} \leqslant \ldots \leqslant A_1^{t^{s-1}} < A_1^{t^s}$$

$$= \{(2k)^k + K\}^{t^s} \leqslant \{2(2k)\}^{t^s}$$

Hence

$$s \geqslant \frac{l_2 x - \log k - \log\log 2k + o(1)}{\log t} = \frac{l_2 x}{\log t} + O(k \log k). \tag{5}$$

* K. Sambasiva Rao, On a particular representation of numbers, *J. I. M. S.* (2) 3 (1939), 262-5.

From (3) and (5), we get

$$S_k(x) \geqslant \frac{l_2 x}{\log t} + M(k) + O(k \log k), \text{ when } x = A_s. \tag{6}$$

3. Let $x_1^k \leqslant x < (x_1+1)^k, x_2^k \leqslant x - x_1^k < (x_2+1)^k$ and so on.

Then $$x - x_1^k < (x_1+1)^k - x^k = x_1^{k-1}\left\{k + \frac{k(k-1)}{2!\,x_1} + \ldots\right\}$$

$$\leqslant (k+1)x_1^{k-1} \leqslant (k+1)x^{\frac{k-1}{k}}, \text{ when } x_1 \geqslant k(k-1), \tag{7}$$

and $$\leqslant 1.3kx_1^{k-1} \leqslant 1.3kx^m, \text{ when } x_1 \geqslant 2k. \tag{8}$$

By applying the above repeatedly, we get

$$x - x_1^k - \ldots - x_s^k \leqslant (k+1)^k x^{m^s}, \text{ when } x_s \geqslant k(k-1), \tag{9}$$

and $$\leqslant (1.3k)^k x^{m^s}, \text{ when } x_s \geqslant 2k. \tag{10}$$

If $$s_1 = \left[\frac{l_2 x - l_1 k - l_2 k}{\log t}\right] + 1, \; (k+1)^k x^{m^{s_1}} \leqslant \{k(k+1)\}^k, \tag{11}$$

and if $$s_2 = \left[\frac{l_1 k + l_2 k + 1}{\log t}\right] + 1, \; (1.3k)^k \left(\{k(k+1)\}\right)^{m^{s_2}} \leqslant (2k)^k, \tag{12}$$

So, if $s = s_1 + s_2$,

$$x - x_1^k - \ldots - x_s^k \leqslant (2k)^k.$$

Therefore

$$S_k(x) \leqslant s_1 + s_2 + M(k)$$

$$\leqslant \frac{l_2 x}{\log t} + M(k) + O(kl_2 k). \tag{13}$$

From (6) and (13), the theorem follows.

It may be observed that

$$M(k) = 2^k + (\tfrac{3}{2})^k + O(\tfrac{4}{3})^k.$$

$$= 2^k + O(\tfrac{3}{2})^k = O(2^k).$$

So the theorem can be stated thus:

$$\text{Max } S_k(x) = \frac{l_2 x}{\log t} + M(k) + O(k \log k)$$

$$= \frac{l_2 x}{\log t} + 2^k + O(\tfrac{3}{2})^k$$

$$= \frac{l_2 x}{\log t} + O(2^k).$$

SELF-RECIPROCAL FUNCTIONS INVOLVING LAGUERRE POLYNOMIALS

By BRIJ MOHAN, *Benares Hindu University.*

[Received 15 March 1939]

1. The object of this note is to prove that the function

$$x^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}L_n^{\frac{1}{2}\nu-\frac{1}{2}}(\tfrac{1}{2}x^2)$$

is self-reciprocal for J_ν transforms, where n is a positive integer.

Adopting a similar procedure, I also obtain the result that the function

$$x^{\nu+2n+\frac{1}{2}}e^{-\frac{1}{2}x^2}L_n^{n+\nu}(\tfrac{1}{2}x^2),$$

where n is a positive integer, is self-reciprocal for J_ν transforms, a theorem proved otherwise by Howell(4).

I shall say that a function is R_ν if it is self-reciprocal for J_ν transforms.

I shall make use of the following result of Hardy and Titchmarsh(3):—

A function $f(x)$ is R_ν if it is given by one of the formulae

$$\text{(i)} \quad f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\psi(s)x^{-s}ds, \qquad (1.1)$$

where $0 < c < 1$,

and $$\psi(s) = \psi(1-s); \qquad (1.2)$$

$$\text{(ii)} \quad f(x)=\frac{x^{\frac{1}{2}-\nu}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}sx^2}s^{-\frac{1}{2}(\nu+1)}\mu(s)ds,$$

where c is positive,

and $$\mu(s)=\mu\left(\frac{1}{s}\right).$$

2. Sonine(6) has given the formula

$$\int_0^\infty e^{-(s+1)x}x^\alpha L_n^\alpha(x)dx=\frac{\Gamma(n+\alpha+1)}{n!}\frac{s^n}{(1+s)^{n+\alpha+1}},$$

where $R(s)>-1$.

Putting $\alpha = \frac{1}{2}\nu - \frac{1}{2}$, we get

$$\int_0^\infty e^{-sx} e^{-x} x^{\frac{1}{2}\nu-\frac{1}{2}} L_n^{\frac{1}{2}\nu-\frac{1}{2}}(x)\,dx = \frac{\Gamma(n+\frac{1}{2}\nu+\frac{1}{2})}{n!} \frac{s^n}{(1+s)^{n+\frac{1}{2}\nu+\frac{1}{2}}}.$$

Therefore, by a form of Mellin's Inversion formula[2],

$$e^{-x} x^{\frac{1}{2}\nu-\frac{1}{2}} L_n^{\frac{1}{2}\nu-\frac{1}{2}}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sx} \frac{\Gamma(n+\frac{1}{2}\nu+\frac{1}{2})}{n!} \frac{s^n}{(1+s)^{n+\frac{1}{2}\nu+\frac{1}{2}}} ds,$$

i.e. $e^{-\frac{1}{2}x^2} x^{-\frac{1}{2}} L_n^{\frac{1}{2}\nu-\frac{1}{2}}(\frac{1}{2}x^2)$

$$= \frac{2^{\frac{1}{2}\nu-\frac{1}{2}}\Gamma(n+\frac{1}{2}\nu+\frac{1}{2})}{n!} \frac{x^{\frac{1}{2}-\nu}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}sx^2} s^{-\frac{1}{2}(\nu+1)} \left(\frac{s}{1+s}\right)^{n+\frac{1}{2}\nu+\frac{1}{2}} ds.$$

The right side of this equation is of the same form as (1.2) with

$$\mu(s) = \frac{2^{\frac{1}{2}\nu-\frac{1}{2}}\Gamma(n+\frac{1}{2}\nu+\frac{1}{2})}{n!} \left(\frac{s}{1+s}\right)^{n+\frac{1}{2}\nu+\frac{1}{2}}$$

Hence the function

$$x^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^{\frac{1}{2}\nu-\frac{1}{2}}(\frac{1}{2}x^2)$$

is R_ν for positive integral values of n.

3. Howell[5] has given the formula

$$\int_0^\infty x^{\alpha+p-1} e^{-x} L_n^\alpha(x)\,dx = \frac{\Gamma(p+\alpha)\,\Gamma(n+1-p)}{n!\,\Gamma(1-p)}.$$

Putting $p = \frac{1}{4} - \frac{1}{2}\nu + \frac{1}{2}s$, $\alpha = \frac{1}{2}\mu + \frac{1}{2}\nu$, $n = \frac{1}{2}\mu - \frac{1}{2}\nu$, we get

$$\int_0^\infty x^{\frac{1}{2}s+\frac{1}{2}\mu-\frac{3}{4}} e^{-x} L_{\frac{1}{2}\mu-\frac{1}{2}\nu}^{\frac{1}{2}\mu+\frac{1}{2}\nu}(x)\,dx = \frac{\Gamma(\frac{1}{4}+\frac{1}{2}\mu+\frac{1}{2}s)\,\Gamma(\frac{3}{4}+\frac{1}{2}\mu-\frac{1}{2}s)}{(\frac{1}{2}\mu-\frac{1}{2}\nu)!\,\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)}.$$

Hence

$$\int_0^\infty y^{\mu+s-\frac{1}{2}} e^{-\frac{1}{2}y^2} L_{\frac{1}{2}\mu-\frac{1}{2}\nu}^{\frac{1}{2}\mu+\frac{1}{2}\nu}(\tfrac{1}{2}y^2)\,dy$$

$$= 2^{-\frac{3}{4}+\frac{1}{2}\mu+\frac{1}{2}s} \frac{\Gamma(\frac{1}{4}+\frac{1}{2}\mu+\frac{1}{2}s)\,\Gamma(\frac{3}{4}+\frac{1}{2}\mu-\frac{1}{2}s)}{(\frac{1}{2}\mu-\frac{1}{2}\nu)!\,\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)}.$$

Hence, using another form of Mellin's Inversion formula[1],

$y^{\mu+\frac{1}{2}} e^{-\frac{1}{2}y^2} L_{\frac{1}{2}\mu-\frac{1}{2}\nu}^{\frac{1}{2}\mu+\frac{1}{2}\nu}(\frac{1}{2}y^2)$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s+\frac{1}{2}\mu-\frac{3}{4}} \frac{\Gamma(\frac{1}{4}+\frac{1}{2}\mu+\frac{1}{2}s)\,\Gamma(\frac{3}{4}+\frac{1}{2}\mu-\frac{1}{2}s)}{(\frac{1}{2}\mu-\frac{1}{2}\nu)!\,\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)} y^{-s} ds$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\psi_1(s) y^{-s} ds, \tag{3.1}$$

where

$$\psi_1(s) = \frac{2^{\frac{1}{2}\mu-\frac{3}{4}}}{(\frac{1}{2}\mu-\frac{1}{2}\nu)!} \frac{\Gamma(\frac{1}{4}+\frac{1}{2}\mu+\frac{1}{2}s)\,\Gamma(\frac{3}{4}+\frac{1}{2}\mu-\frac{1}{2}s)}{\Gamma(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)\,\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)}.$$

The integrand in (3.1) is of the same form as in (1.1) and $\psi_1(s)$ obviously satisfies the equation (1.2).

Hence the function

$$x^{\mu+\frac{1}{2}} e^{-\frac{1}{2}x^2} L_{\frac{1}{2}\mu-\frac{1}{2}\nu}^{\frac{1}{2}\mu+\frac{1}{2}\nu}(\tfrac{1}{2}x^2)$$

is R_ν, where $\frac{1}{2}\mu-\frac{1}{2}\nu$ is a positive integer.

In other words, the function

$$x^{\nu+2n+\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^{n+\nu}(\tfrac{1}{2}x^2)$$

is R_ν for positive integral values of n.

REFERENCES

(1) G. H. HARDY: On Mellin's Inversion Formula, *Messenger of Math.* 47 (1918), 178-84.

(2) ———: Further notes on Mellin's Inversion Formula, *Ibid.* 50 (1921), 165-71.

(3) ——— AND E. C. TITCHMARSH: Self-Reprocal functions. *Quart. Jour. Math.* (Oxford) 1 (1930), 196-231, §§ 1.4, 1.5.

(4) W. T. HOWELL: A note on Laguerre Polynomials, *Phil. Mag.* VII 23 (1937), 807-11.

(5) ———: On products of Laguerre Polynomials, *Ibid.* 24 (1937), 396-405 (399, 403).

(6) N. SONINE: *Math. Ann.* XVI (1880), 1-80.

A BESSEL FUNCTION IN TERMS OF INCOMPLETE GAMMA FUNCTIONS

By W. F. KIBBLE, *Madras Christian College.*

[Received 26 April 1939]

1. Introduction.
2. The Incomplete Gamma Function.
3. The finite form of the series.
4. The general form of the series.
5. Convergence of the series.
6. Equating the series to a Bessel function.
7. Generalised binomial coefficients.
8. Lemma: Evaluation of a determinant.
9. Uniqueness of the solution.
10. The corresponding series with (complete) Gamma functions.
11. Conclusion.

1. *Introduction.* It has been shown by Hadamard* that it is possible to modify the asymptotic expansion† of the Bessel function of imaginary argument

$$\frac{e^z}{(2\pi z)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\binom{\nu-\frac{1}{2}}{m}\frac{\Gamma(\nu+m+\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})}(-2z)^{-m}\sim\frac{e^{-\nu\pi i}I_\nu(z)-e^{\nu\pi i}I_{-\nu}(z)}{e^{-\nu\pi i}-e^{\nu\pi i}} \quad (1)$$

so that it becomes a convergent series with a negligible remainder term. The modification which he gives for functions of zero order‡, and which is given by Watson§ for functions of any order greater than $-\frac{1}{2}$, is formed by replacing the Gamma function $\Gamma(\nu+m+\frac{1}{2})$ by the Incomplete Gamma function $\gamma(\nu+m+\frac{1}{2}, 2z)$.

The object of this paper is to find out what other similar expansions can be found as solutions of Bessel's equation.

It is shown that if the real part of ν is greater than $-\frac{1}{2}$, then a convergent series for $I_\nu(z)$ is

$$I_\nu(z)=\frac{e^z(2z)^a}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\sum_{m=-\infty}^{\infty}\binom{\nu-\frac{1}{2}}{m-a-\frac{1}{2}}(-2z)^{-m}\gamma(\nu+m-a, 2z), \quad (2)$$

* *Trans. Amer. Math. Soc.* 3 (1902), 421-2.

† H. Hankel, *Math. Ann.* 1 (1869), 491-5.

§ G. N. Watson, *Theory of Bessel Functions*, (1922), chap. VII.

‡ *Bull. Soc. Math. France* XXXVI (1908), 77-85.

§ *loc. cit.* pp. 204-5.

where $\binom{\nu-\frac{1}{2}}{m-a-\frac{1}{2}}$ now denotes the 'generalised binomial coefficient':

$$\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu-m+a+1)\,\Gamma(m-a+\frac{1}{2})}. \tag{3}$$

This reduces to Watson's result when $a+\frac{1}{2}$ is any integer. When $\nu+a$ is any integer it reduces to

$$I_\nu(z)=\frac{e^z(2z)^{-\nu}}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\sum_{m=0}^{\infty}\binom{\nu-\frac{1}{2}}{m}(-2z)^m\gamma(2\nu-m,\,2z), \tag{4}$$

and the series formed by replacing $\gamma(2\nu-m,\,2z)$ in this series by $\Gamma(2\nu-m)$ is convergent for all values of ν and represents a Bessel function.

It is shown in **10** that when in (2) we replace $\gamma(\nu+m-a,2z)$ by $\Gamma(\nu+m+a)$, we obtain an asymptotic expansion of which that convergent series and the expansion (1) are special cases.

The question of more general solutions of the type

$$z^a e^{pz}\sum_{m=-\infty}^{\infty} c_m(kz)^{-m}\gamma(\mu+m,\,kz) \tag{5}$$

is discussed in **4**. It is shown that μ must be such that

$$(\mu+a)^2=\nu^2, \tag{6}$$

and it is shown in **9**, that for any given a and p there cannot, for one of the values of μ, be more than one series of the type (5) (with k determined as in § 4. 7), which is a convergent solution of Bessel's differential equation.

2. *The Incomplete Gamma Function.* When z is real and positive and s has a positive real part, the definition of the Incomplete Gamma Function is

$$\gamma(s,\,z)=\Gamma_z(s)=\int_0^z t^{s-1}e^{-t}dt.$$

Successive integration by parts gives

$$\gamma(s,z)=$$
$$z^s e^{-z}\left\{\frac{1}{s}+\frac{z}{s(s+1)}+\frac{z^2}{s(s+1)(s+2)}+\ldots+\frac{z^m}{s(s+1)\ldots(s+m)}\right\}$$
$$+\frac{\int_0^z t^{s+m-1}e^{-t}dt}{s(s+1)\ldots(s+m)}, \tag{7}$$

in which the series is absolutely convergent for all z, and the remainder tends to zero as m tends to infinity. The series (7) is

therefore taken as the definition of the function for complex values of z and s. The difference between the Gamma function and the incomplete Gamma function can be given by an integral* which converges for all z, viz.

$$\int_z^{+\infty} t^{s-1}e^{-t}dt. \tag{8}$$

In this paper we consider series of the type:

$$\sum_{m=-\infty}^{\infty} c_m(kz)^{-m}\gamma(m+\mu, kz), \tag{9}$$

and since the series (7) is absolutely and uniformly convergent, we can substitute and rearrange in powers of z:

$$e^{-kz}(kz)^{\mu}\left\{\sum\frac{c_m}{\mu+m}+\sum\frac{c_m z}{(\mu+m)(\mu+m+1)}\right.$$
$$\left.+\sum\frac{c_m z^2}{(\mu+m)(\mu+m+1)(\mu+m+2)}+\ldots\right\}. \tag{10}$$

Each of these series is convergent if the first is convergent and thus the necessary and sufficient condition that the series (9) should converge is that $\Sigma\, c_m/(\mu+m)$ should converge. The principal part in the neighbourhood of the origin of the series (9) is

$$(kz)^{\mu}\sum_{-\infty}^{\infty}\frac{c_m}{\mu+m}.$$

Note: The notation $\gamma(s, z)$ for the incomplete Gamma function is used in Whittaker and Watson's *Modern Analysis* and in Watson's *Bessel Functions*. The notation $\Gamma_z(s)$ is adopted in the tables of the function in *Tracts for Computors*.

3. *A finite series.*

3.1. It is of interest to consider first the case in which the series consists of a finite number of terms only; the analys is is then simpler than in the general case.

We shall therefore consider for what values of the constants $a, p, k, n, \mu, c_0, c_1, c_2, \ldots, c_n$, the function $F(z)$ defined as

$$z^a e^{pz}\sum_{m=0}^{n} c_m(kz)^{-m}\gamma(m+\mu, kz) \tag{11}$$

* L. N. Milne-Thomson, *Finite Differences*, where the integral (8) is also called an incomplete Gamma function.

is a solution of the differential equation

$$\frac{d^2F}{dz^2}+\frac{1}{z}\frac{dF}{dz}-\left(K^2+\frac{N^2}{z^2}\right)F=0, \tag{12}$$

where n is a positive integer, k is not zero, μ is not zero and not a negative integer, and otherwise the constants may be any complex numbers.

3.2. Substituting in (11) for $\gamma(m+\mu, kz)$ from the series (7), we have

$$F(z)=z^a e^{(p-k)z}\sum_{m=0}^{n}\sum_{t=0}^{\infty} c_m\Gamma(m+\mu)\frac{(kz)^{\mu+t}}{\Gamma(m+\mu+t+1)},$$

the rearrangement in the order of the powers of z being permissible since the series (7) is absolutely convergent. Denote the coefficient of $z^{a+\mu+t}e^{(p-k)z}$ in $F(z)$ by

$$a_t=\sum_{m=0}^{n} c_m\frac{\Gamma(m+\mu)}{\Gamma(m+\mu+t+1)}k^{\mu+t}, \text{ if } t\geqslant 0,\ a_t=0 \text{ if } t<0. \tag{13}$$

Then

$$\left.\begin{aligned}
F(z)&=\sum_{t=0}^{\infty} e^{(p-k)z}z^{a+\mu+t}a_t\\
\frac{\partial F}{\partial z}&=\sum_{t=-1}^{\infty}\left\{(p-k)a_t+(a+\mu+t+1)a_{t+1}\right\}e^{(p-k)z}z^{a+\mu+t}\\
\frac{\partial^2 F}{\partial z^2}&=\sum_{t=-2}^{\infty}\left\{(p-k)^2a_t+2(p-k)(a+\mu+t+1)a_{t+1}\right.\\
&\qquad\left.+(a+\mu+t+1)(a+\mu+t+2)a_{t+2}\right\}e^{(p-k)z}z^{a+\mu+t}
\end{aligned}\right\} . \tag{14}$$

Substituting from equations (14) in equation (12), omitting the common factor $e^{(p-k)z}$, and equating coefficients of powers of z, we obtain the set of conditions required.

$$\begin{aligned}
&\left\{(p-k)^2-K^2\right\}a_t+(p-k)\left\{2(a+\mu+t+1)+1)\right\}a_{t+1}\\
&\quad+\left\{(a+\mu+t+1)(a+\mu+t+2)+(a+\mu+t+2)-N^2\right\}a_{t+2}=0,\\
&\qquad\qquad t=-2,\,-1,\,0,\,1,\,2,\ldots
\end{aligned} \tag{15}$$

Substituting for a_t, a_{t+1}, a_{t+2}, omitting the common factor $k_g^{\mu+t}$, this becomes for $t = 0, 1, 2, \ldots,$

$$\left\{ (p-k)^2 - K^2 \right\} \sum_{m=0}^{n} c_m \frac{\Gamma(m+\mu)}{\Gamma(m+\mu+t+1)}$$

$$+2(p-k)(a+\mu+t+\tfrac{3}{2}) \sum_{m=0}^{n} k c_m \frac{\Gamma(m+\mu)}{\Gamma(m+\mu+t+2)}$$

$$+\left\{ (a+\mu+t+2)^2 - N^2 \right\} \sum_{m=0}^{n} k^2 c_m \frac{\Gamma(m+\mu)}{\Gamma(m+\mu+t+3)} = 0.$$

Multiplying throughout by $\Gamma(t+\mu+n+3)$, we obtain an equation of degree $n+2$ in t:

$$\left\{ (p-k)^2 - K^2 \right\} \Sigma c_m \Gamma(m+\mu).(m+\mu+t+1)(m+\mu+t+2) \ldots (n+\mu+t+2)$$
$$+2(p-k)k(a+\mu+t+1) \Sigma c_m \Gamma(m+\mu).(m+\mu+t+2)(m+\mu+t+3) \ldots (n+\mu+t+2)$$
$$+k^2 \left\{ (a+\mu+t+2)^2 - N^2 \right\} \Sigma c_m \Gamma(m+\mu).(m+\mu+t+3) \ldots (n+\mu+t+2) = 0. \quad (16)$$

Equating to zero the coefficients in (16) we obtain $n+3$ equations, which together with the two equations (15) for which $t = -1$ and $t = -2$, form the necessary and sufficient set of conditions that the function (11) should satisfy the equation (12).

3.3. From the coefficient of t^{n+2} we obtain, assuming $c_0 \neq 0$,

$$\left\{ (p-k)^2 - K^2 \right\} + 2(p-k)k + k^2 = 0;$$

i.e.

$$K^2 = p^2; \quad (17)$$

and the coefficient of t^{n+1},

$$[(p-k)^2 - K^2]\left[c_0 \Gamma(\mu) . \frac{n+2}{2}(n+2\mu+3) + c_1 \Gamma(\mu+1) \right]$$
$$+2(p-k)k\left\{ c_0 \Gamma(\mu) \left[a+\mu+\frac{3}{2} + \frac{n+1}{2}(n+2\mu+4) \right] + c_1 \Gamma(\mu+1) \right\}$$
$$+k^2 \left\{ c_0 \Gamma(\mu) \left[2(a+\mu+2) + \frac{n}{2}(n+2\mu+5) \right] + c_1 \Gamma(\mu+1) \right\} = 0,$$

which reduces, by (17) to

$$(k^2 - 2kp)(2\mu+3) + 2(p-k)k[a+\mu+\tfrac{3}{2}+\tfrac{1}{2}(2\mu+4)]$$

i.e. $$2pk(2a+1)=0;$$
so that either
$$p=0, \quad \text{or} \quad 2a+1=0. \tag{18}$$

3.31. If $p=0$, then $K=0$ and the equations (21) below have no solution in which $b_0, b_1, b_2, \ldots, b_n$ are not all zero unless $\nu^2=a^2$, and then $b_1=b_2=b_3=\ldots=b_n=0$, so that if $K=0$ there is no solution in the form of a series such as (11). We put $K=1$, so that (11) becomes Bessel's equation, in the modified form for a purely imaginary argument. And (18) gives $2a+1=0$, while (17) gives $p=\pm 1$.

3.4. To reduce the $n+1$ equations obtained by equating to zero the coefficients of $t^n, t^{n-1}, \ldots$ in (16), put
$$b_m = c_m\Gamma(m+\mu) \tag{19}$$
and let $S(r,s)$ denote the sum of all products of r of the numbers $\mu+s+1, \mu+s+2, \ldots, \mu+n+2$, so that
$$S(r,s+1) = S(r,s)-(\mu+s+1).S(r-1,s+1),$$
$$S(0,s)=1, \qquad s=0,1,2,\ldots,n+1. \tag{20}$$

Then the coefficient in (16) of t^{n-j} is
$$(k^2-2kp)[b_0S(j+2,0)+b_1S(j+1,1)+\ldots+b_{j+2}S(0,j+2)]$$
$$+2(p-k)k(\mu+1)[b_0S(j+1,1)+b_1S(j,2)+\ldots+b_{j+1}S(0,j+2)]$$
$$+2(p-k)k[b_0S(j+2,1)+b_1S(j+1,2)+\ldots+b_{j+2}S(0,j+3)]$$
$$+k^2[(\mu+\tfrac{3}{2})^2-\nu^2][b_0S(j,2)+b_1S(j-1,3)+\ldots+b_jS(0,j+2)]$$
$$+2k^2(\mu+\tfrac{3}{2})[b_0S(j+1,2)+b_1S(j,3)+\ldots+b_{j+1}S(0,j+3)]$$
$$+k^2[b_0S(j+2,2)+b_1S(j+1,3)+\ldots+b_{j+2}S(0,j+4)]=0.$$

Simplifying this by equations (20), we have, for $j=0,1,2,\ldots,n$
$$2(p-k)(\mu+1)[b_0S(j+1,1)+b_1S(j,2)+\ldots+b_{j+1}S(0,j+2)]$$
$$+(k-2p)[(\mu+1)b_0S(j+1,1)+(\mu+2)b_1S(j,2)+\ldots$$
$$+(\mu+j+2)b_{j+1}S(0,j+2)]$$
$$+k[(\mu+\tfrac{3}{2})^2-\nu^2][b_0S(j,2)+b_1S(j-1,3)+\ldots+b_jS(0,j+2)]$$
$$+2k(\mu+\tfrac{3}{2})[b_0S(j+1,2)+b_1S(j,3)+\ldots+b_{j+1}S(0,j+3)]$$
$$-k[(\mu+2)b_0S(j+1,2)+(\mu+3)b_1S(j,3)+\ldots$$
$$+(u+j+3)b_{j+1}S(0,j+3)]=0. \tag{21}$$

The coefficient of b_m is

$$-2pm\, S(j-m+1,\, m+1)+k\Big\{(\mu-m+1)\,[S(j-m+1, m+2) \\ -S(j-m+1,\, m+1)]+(\mu^2+3\mu+\tfrac{9}{4}-\nu^2)\; S(j-m,\, m+2)\Big\}$$
$$=-2pm\; S(j-m+1,\, m+1)+k(m^2+m+\tfrac{1}{4}-\nu^2)S(j-m,\, m+2),$$

so that the determinant obtained by eliminating $b_0, b_1, b_2, \ldots, b_n$, from the $n+1$ equations (21) is a determinant in which the element in the rth row and the sth column is

$$k\Big\{(s-\tfrac{1}{2})^2-\nu^2\Big\}S(r-s,\, s+1)-2p(s-1)S(r-s+1,\, s).$$

Each element for which $s > r+1$ is zero. It can be seen that the determinant vanishes if $\nu^2 = (s-\tfrac{1}{2})^2$, where $s = 1, 2, \ldots, n$, or $n+1$.

The determinant is of degree $n-1$ in ν^2, and, equating it to zero, we see that the condition that the equations (21) should have a non-zero solution is that ν^2 should have one of the values $(\tfrac{1}{2})^2$, $(\tfrac{3}{2})^2$, $(\tfrac{5}{2})^2, \ldots, (n+\tfrac{1}{2})^2$. If however $\nu^2 = (m+\tfrac{1}{2})^2$, where $m < n$, the solution of (21) is one in which $b_j = 0$, for all $j \geqslant m+1$. So the only solution which does not in fact correspond to a lower value of n, is $\nu^2 = (n+\tfrac{1}{2})^2$.

3.5. Solution of (21) now gives

$$b_m = \frac{k}{2p}\,\frac{(m-\tfrac{1}{2})^2-\nu^2}{m}\, b_{m-1} = \frac{k}{2p}\,\frac{(m-n-1)(m+n)}{m}\, b_{m-1}$$
$$= \left(-\frac{k}{2p}\right)^m b_0\, \frac{n!}{(n-m)!\,m!}\,\frac{(n+m)!}{n!}. \qquad (23)$$

3.6. We have now found the values which

$$a,\ p,\ n,\ \frac{b_1}{b_0},\ \frac{b_2}{b_0}, \ldots, \frac{b_n}{b_0}$$

must have in order that those of the equations (15) for which $t = 0, 1, 2, 3, \ldots$, may be satisfied. The equations (15) for $t = -1$ and $t = -2$ will give k and μ.

The equation for $t = -2$ is

$$(a+\mu-1)(a+\mu)+(a+\mu)-\nu^2 = (a+\mu)^2-\nu^2 = 0. \qquad (24)$$

Hence $(\mu-\frac{1}{2})^2 = (\mu+a)^2 = \nu^2 = (n+\frac{1}{2})^2.$

A negative integral or zero value for μ is not permissible, and therefore $n = \mu - 1$.

Then, from (23) and (19), we have

$$c_m = b_m/(n+m)! = \left(-\frac{k}{2p}\right)^m c_0 \frac{n!}{(n-m)!\,m!}.$$

It is convenient to put $c_0 = 1$, so that

$$c_m = \binom{n}{m}\left(-\frac{k}{2p}\right)^m. \tag{25}$$

3.7. Now the equation (15) for $t = -1$ is

$$(p-k)a_0 + a_1 = 0, \tag{26}$$

and

$$a_0 = \sum_{m=0}^{n}\binom{n}{m}\left(-\frac{k}{2p}\right)^m \frac{k^\mu}{m+\mu},$$

$$a_1 = \sum\binom{n}{m}\left(-\frac{k}{2p}\right)^m \frac{k^{\mu+1}}{(m+\mu+1)\,(m+\mu)}$$

$$= \sum\binom{n}{m}\left(-\frac{k}{2p}\right)^m k^{\mu+1}\left[\frac{1}{m+\mu} - \frac{1}{m+\mu+1}\right]$$

Since

$$\sum\binom{n}{m}\left(-\frac{k}{2p}\right)^{m+\mu}\frac{(-1)^{-\mu}}{m+\mu} = \sum\binom{n}{m}\int_0^{k/2p}(-1)^m u^{m+\mu-1}\,du$$

$$= \int_0^{k/2p}(1-u)^n u^{\mu-1}du, \quad \text{the incomplete Beta function,}$$

we shall denote

$$\sum\binom{n}{m}\left(-\frac{k}{2p}\right)^{m+\mu}\frac{(-1)^{-\mu}}{m+\mu} \quad \text{by } \beta\left(n+1, \mu; \frac{k}{2p}\right)$$

and equation (26) can be written as

$$\beta\left(n+1, \mu; \frac{k}{2p}\right) = 2\beta\left(n+1, \mu+1; \frac{k}{2p}\right) \tag{27}$$

Now

$$\frac{\partial}{\partial(k/2p)}\left[\beta\left(n+1, \mu; \frac{k}{2p}\right) - 2\beta\left(n+1, \mu+1; \frac{k}{2p}\right)\right]$$

$$= \left(1-\frac{k}{2p}\right)^n \left(\frac{k}{2p}\right)^{\mu-1} \left(1-\frac{k}{p}\right)$$

$$= x.\frac{(1-x^2)^n}{4^n} \quad \text{if } x = 1-\frac{k}{p}, \text{ since } n = \mu-1.$$

Integrating with respect to x we see that

$$\beta\left(n+1, n+1; \frac{k}{2p}\right) - 2\beta\left(n+1, n+2; \frac{k}{2p}\right)$$

is proportional to $(1-x^2)^{n+1}$, for it is zero when $k = 0$, i.e. when $x = 1$. Thus the only non-zero value of k for which (27) is satisfied is given by

$$x = -1, \; k = 2p. \tag{28}$$

3.8. We see thus that

$$z^{-\frac{1}{2}} e^{z} \sum_{m=0}^{n} (-1)^m \binom{n}{m} (2z)^{-m} \gamma(m+n+1, 2z)$$

and (29)

$$z^{-\frac{1}{2}} e^{-z} \sum_{m=0}^{n} (-1)^m \binom{n}{m} (-2z)^{-m} \gamma(m+n+1, -2z)$$

are the only solutions in this form of the modified Bessel equation of order $n+\frac{1}{2}$, and that if the order is not half an odd integer, no solution in this finite form is possible.

These series can be written in the reverse order, and we have then a series in ascending powers of z:

$$z^{-n-\frac{1}{2}} e^{\pm z} \sum_{m=0}^{n} (-1)^m \binom{n}{m} (\pm 2z)^m \gamma(2n-m+1, \pm 2z). \tag{30}$$

We shall show in the next section that both (29) and (30) give convergent series when n is not a positive or negative integer, and in (30), is not half a positive integer, and that these series are solutions of the equation of order $n+\frac{1}{2}$, provided that the real part of n is greater than -1.

4. *The general form of the series.*

4.1. We shall now consider the conditions that

$$F(z) = z^a e^{pz} \sum_{m=-\infty}^{\infty} c_m (kz)^{-m} \gamma(m+\mu, kz) \tag{31}$$

should be a formal solution of the modified Bessel equation

$$\frac{d^2F}{dz^2}+\frac{1}{z}\frac{dF}{dz}-\left(K^2+\frac{\nu^2}{z^2}\right)F=0 \tag{32}$$

postponing consideration of its convergence to **5**.

Differentiating (31),

$$\frac{dF}{dz}=z^a e^{pz}\sum c_m(a-m)k^{-m}z^{-m-1}\gamma(m+\mu, kz)$$

$$+pz^a e^{pz}\sum c_m k^{-m}z^{-m}\gamma(m+\mu, kz)+z^a e^{pz}\sum c_m k^{\mu}z^{\mu-1}e^{-kz},$$

$$\frac{d^2F}{dz^2}=z^a e^{pz}\sum c_m(kz)^{-m}\left[\frac{(a-m)(a-m-1)}{z^2}+\frac{2(a-m)p}{z}+p^2\right]$$

$$\times\gamma(m+\mu, kz)+z^{a+\mu-1}e^{(p-k)z}k^{\mu}\sum c_m\left(\frac{a-m}{z}+p\right)$$

$$+\left(\frac{a+\mu-1}{z}+p-k\right)z^{a+\mu-1}e^{(p-k)z}k^{\mu}\sum c_m.$$

Substituting in (32), we get

$$z^a e^{pz}\sum c_m(kz)^{-m}\left[\frac{(a-m)^2-\nu^2}{z^2}+\frac{2(a-m)p+p}{z}+p^2-K^2\right]$$

$$\times\gamma(m+\mu, kz)+z^{a+\mu-1}e^{(p-k)z}k^{\mu}\sum c_m\left(\frac{2a-m+\mu}{z}+2p-k\right)=0. \tag{33}$$

Substituting for $\gamma(m+\mu, kz)$ from the series (6), we may rearrange according to powers of z, since that series is absolutely convergent. Equation (33) becomes

$$\sum c_m\left[\frac{(a-m)^2-\nu}{z^2}+\frac{(2a+1-2m)p}{z}+p^2-K^2\right]\times$$

$$\left[\frac{1}{m+\mu}+\frac{kz}{(m+\mu)(m+\mu+1)}+\frac{(kz)^2}{(m+\mu)(m+\mu+1)(m+\mu+2)}+\dots\right]$$

$$+\sum c_m\left[\frac{2a+\mu+m}{z^2}+\frac{2p-k}{z}\right]=0, \tag{34}$$

omitting the factor $z^{a+\mu}e^{(p-k)z}k^{\mu}$. Equating to zero the coefficient in (34) of $z^{-2}, z^{-1}, z^0, z, z^2, \dots$, we have

$$\sum c_m\left[\frac{(a-m)^2-\nu^2}{m+\mu}+2a+\mu-m\right]=0 \tag{35}$$

$$\sum c_m\left[\frac{(a-m)^2-\nu^2}{(m+\mu)(m+\mu+1)}k+\frac{(2a+1-2m)p}{m+\mu}+2p-k\right]=0 \quad (36)$$

and

$$\sum c_m\Gamma(m+\mu)\left[\frac{(a-m)^2-\nu^2}{\Gamma(m+\mu+s+3)}k^{s+2}+\frac{(2a+1-2m)p}{\Gamma(m+\mu+s+2)}k^{s+1}\right.$$
$$\left.+\frac{(p^2-K^2)}{\Gamma(m+\mu+s+1)}k^s\right]=0, \quad (37)$$
$$s=0,1,2,3,\ldots .$$

Equation (35) reduces to

$$\left[\nu^2-(\mu+a)^2\right]\sum\frac{c_m}{m+\mu}=0, \quad (38)$$

and (36) to

$$\left[k[(a+\mu)^2-\nu^2]+p(2a+1+2\mu)\right]\sum\frac{c_m}{m+\mu}$$
$$-k[(a+\mu+1)^2-\nu^2]\sum\frac{c_m}{m+\mu+1}=0, \quad (39)$$

and (37) can be written, using partial fractions, as

$$\sum c_m k^2[(a-m)^2-\nu^2]\left[\frac{1}{m+\mu}-\frac{s+2}{m+\mu+1}+\frac{\binom{s+2}{2}}{m+\mu+2}-\frac{\binom{s+2}{3}}{m+\mu+3}\right.$$
$$\left.+\ldots\pm\frac{1}{m+\mu+s+2}\right]\cdot\frac{1}{(s+2)!}$$
$$\pm\sum c_m kp(2a+1-2m)\left[\frac{1}{m+\mu}-\frac{s+1}{m+\mu+1}+\frac{\binom{s+1}{2}}{m+\mu+2}-\frac{\binom{s+1}{3}}{m+\mu+3}\right.$$
$$\left.+\ldots\mp\frac{1}{m+\mu+s+1}\right]\cdot\frac{1}{(s+1)!}$$
$$+\sum c_m(p^2-K^2)\left[\frac{1}{m+\mu}-\frac{s}{m+\mu+1}+\frac{\binom{s}{2}}{m+\mu+2}-\frac{\binom{s}{3}}{m+\mu+3}\right.$$
$$\left.+\ldots\pm\frac{1}{m+\mu+s}\right]\cdot\frac{1}{s!}=0.$$

That is, if we assume from (38) that $\nu^2=(\mu+a)^2$,

$$\sum c_m k^2\left[-\frac{2\mu+2a+1}{m+\mu+1}\cdot\frac{1}{(s+1)!}+\frac{2(2\mu+2a+2)}{m+\mu+2}\cdot\frac{1}{2!s!}\right.$$
$$\left.-\frac{3(2\mu+2a+3)}{m+\mu+3}\cdot\frac{1}{3!(s-1)!}+\cdots\right]$$

$$+\sum c_m kp\left[\frac{2\mu+2a+1}{m+\mu}\cdot\frac{1}{(s+1)!}-\frac{2\mu+2a+3}{m+\mu+1}\cdot\frac{1}{s!}\right.$$
$$\left.+\frac{2\mu+2a+5}{m+\mu+2}\cdot\frac{1}{2!(s-1)!}-\ldots\right]$$
$$+\sum c_m(p^2-K^2)\left[\frac{1}{m+\mu}\cdot\frac{1}{s!}-\frac{1}{m+\mu+1}\cdot\frac{1}{(s-1)!}+\frac{1}{m+\mu+2}\cdot\frac{1}{2!(s-2)!}\right.$$
$$\left.-\ldots\right]=0,\ s=0,1,2,3,\ldots\,. \tag{40}$$

Using equation (39), and then successively subtracting each equation of (40) from the succeeding equation, we see that the equations (40) are equivalent to

$$\sum c_m k^2\frac{2\mu+2a+j+2}{m+\mu+j+2}-\sum c_m kp\,\frac{2\mu+2a+j+3}{m+\mu+j+1}$$
$$+\sum c_m(p^2-K^2)\frac{j+1}{m+\mu+j}=0,\ j=0,1,2,3\ldots\,. \tag{41}$$

These equations, with (39), form a set of recurrence formulae from which we can determine uniquely the value of Σ_j/Σ_0, where

$$\sum_j=\sum\frac{c_m}{m+\mu+j} \tag{42}$$

for $j=1,2,3,\ldots$.

4.2. If in writing (40) we do not make the assumption that $\nu^2=(\mu+a)^2$ we still obtain a set of recurrence relations for Σ_j, with $\dfrac{(\mu+a+j+2)^2-\nu^2}{j+2}$ instead of $2\mu+2a+j+2$ in the numerator of the first term. (The possibility that $\nu^2=(\mu+a+j)^2$ for some $j>0$ need not be further considered when we see that

$$k^j z^{a+j}e^{pz}\sum c_m(kz)^{-m}\,\gamma(m+\mu-j,\ kz)$$

is another way of writing

$$z^a e^{pz}\sum c_m(kz)^{-m}\,\gamma(m+\mu,\ kz) \tag{43}$$

and the value of a has yet to be found.)

Thus, unless each Σ_j is zero, (38) gives

$$\nu^2=(\mu+a)^2. \tag{44}$$

(The impossibility of a solution in which $\Sigma_j=0$ for all j, is shown below, 9)

4.3. Equation (39) can be included in the set (41), with $j=-1$. We have to solve the equations

$$k^2\Sigma_{j+2}(2N+j+2)-kp\Sigma_{j+1}(2N+2j+3)+(p^2-K^2)\Sigma_j\,(j+1) = 0, \qquad (45)$$

where $j=-1, 0, 1, 2, \ldots$, and N is written for $\mu+a$.

4.4. It may be verified by substitution that the solution is

$$\frac{\Sigma_j}{\Sigma}=\left(\frac{p}{k}\right)^j\left\{1+\frac{j(j-1)}{2}\,\frac{K^2}{(2N+2)p^2}+\frac{j(j-1)(j-2)(j-3)}{2.4}\cdot\frac{K^4}{(2N+2)(2N+4)p^4}+\ldots\right\} \qquad (46)$$

from which we see that if $K=0$, no solution is possible in which $\Sigma_j/\Sigma_{j-1}\to 1$ and $\Sigma_j\to 0$. We shall therefore without loss of generality, put $K=1$. (46) gives the solution of (45) even when $p=0$. The coefficients are the coefficients in the Hermite polynomial, but with positive sign throughout. Binet's result*, that, when $K=p=\frac{1}{2}k$,

$$\Sigma_j/\Sigma_0 = B(N+\tfrac{1}{2}, N+\tfrac{1}{2}+j)/B(N+\tfrac{1}{2}, N+\tfrac{1}{2}),$$

if $N+\frac{1}{2}>0$, may be deduced from (41).

4.5. We shall now assume that Σ_j tends to zero as j tends to infinity, a condition which is satisfied if c_m is bounded, and tends to zero as m tends to $+\infty$ or to $-\infty$. We assume further that Σ_j/Σ_{j-1} tends to 1 as j tends to infinity, a condition which is satisfied if $\sum_{-\infty}^{0} c_m$ and $\sum_{0}^{\infty} c_m$ are convergent, and $c_m=O(1/-m)$ as $m\to-\infty$. If $p=0$, $\Sigma_j=0$ for all odd j. We shall therefore assume $p\neq 0$.

We see from (45) that a necessary condition for Σ_j/Σ_{j-1} to tend to 1 is

$$k^2-2kp+p^2-1=0$$

i.e.

$$k=p\pm 1. \qquad (47)$$

4.6. To find the sufficient condition, we consider the ratio of the $(s+1)$th term in (46) to the sth. This ratio is

$$\frac{(j-2s+2)(j-2s+1)}{2s(2N+2s)}\cdot\frac{1}{p^2}$$

* Whittaker & Watson, *Modern Analysis*, (1920). p. 262.

and the corresponding ratio in the expansion of

$$\left(1+\frac{1}{p}\right)^j+\left(1-\frac{1}{p}\right)^j \tag{48}$$

is

$$\frac{(j-2s+2)(j-2s+1)}{(2s-1)2s}\cdot\frac{1}{p^2}.$$

Now, given any positive number ϵ we can find r and s, functions of j, such that $r<s$, $r\to\infty$, $(s-r)/r\to 0$, as $j\to\infty$, and such that the sum of the terms between the rth and the sth in (48) differs from the total sum by less than ϵ times that sum, and it is evident from the argument of this section that the same functions r and s will serve for the series in (46) as for (48). Now the ratio of a typical term in (46), that is, in the series for Σ_j/Σ_0, to the corresponding term in the series for Σ_{j-1}/Σ_0 is $(p/k)[j/(j-2s)]$, and the corresponding ratio for (48) is $j/(j-2s)$, so that

$$\Sigma_j/\Sigma_{j-1}\sim (p/k)[j/(j-2s)];$$

and

$$\frac{\left(1+\frac{1}{p}\right)^j+\left(1-\frac{1}{p}\right)^j}{\left(1+\frac{1}{p}\right)^{j-1}+\left(1-\frac{1}{p}\right)^{j-1}}\sim\frac{j}{j-2s}.$$

Thus the condition that Σ_j/Σ_{j-1} should tend to 1 is

$$\frac{p}{k}\cdot\frac{\left(1+\frac{1}{p}\right)^j+\left(1-\frac{1}{p}\right)^j}{\left(1+\frac{1}{p}\right)^{j-1}+\left(1-\frac{1}{p}\right)^{j-1}}\to 1 \text{ as } j\to\infty.$$

If $k=p+1$, this condition is (real part of p) >0, and if $k=p-1$, (real part of p) <0. If real part of $p=0$, the condition cannot be satisfied.

4.7. If any function of z is a solution of Bessel's equation, the same function of $-z$ is also a solution, and we need therefore only discuss the values of p with a positive real part, the form of the theorem for a p with a negative real part can then be deduced by changing the sign of z. For p with positive real part, then, the necessary and sufficient condition that Σ_j/Σ_{j-1} should tend to 1 as j tends to infinity is that $k=p+1$.

4.8. Comparing (46) and (48) we see that the ratio of the $(s+1)$th term in Σ_j/Σ_0 to the $(s+1)$th in

$$\left(\frac{p}{k}\right)^j\left\{\left(1+\frac{1}{p}\right)^j+\left(1-\frac{1}{p}\right)^j\right\}$$

is

$$\frac{1.3.5.7.\ldots..(2s-1)}{(2N+2)(2N+4)\ldots\ldots(2N+2s)}$$

and hence

$$\frac{\Sigma_j}{\Sigma_0}\sim\left(\frac{p}{k}\right)^j\left\{\left(1+\frac{1}{p}\right)^j+\left(1-\frac{1}{p}\right)^j\right\}$$

$$\times\frac{1.3.5.\ldots(2s-1)}{(2N+2)(2N+4)\ldots(2N+2s)}$$

$$\sim\frac{1.3.5.\ldots(2s-1)}{(2N+2)(2N+4)\ldots(2N+2s)} \quad (49)$$

where s can be taken to be the s of § 4.6 or the r of that section or any number between them. [We may take s such that the sth term in (46) or (48) is the greatest, so that $s\sim j/(p+1)$.] Hence the necessary and sufficient condition that Σ_j should tend to zero as j tends to infinity is that real part of $\mu+a=$ real part of $N>-\frac{1}{2}$.

4.9. We shall now show that if the numbers c_m are such as to satisfy the recurrence relations

$$(m+\mu-1)(p^2-1)c_m+(2a-2m+3)kpc_{m-1}$$
$$-(\mu+2a-m+2)k^2c_{m-2}=0,\ m=\ldots,-2,-1,0,1,2,\ldots, \quad (50)$$

then the equation (45) is satisfied, $j=-1, 0, 1, 2, \ldots$.

The proof depends on the facts that

$$\sum_{-\infty}^{\infty}c_m=\sum c_{m-1}=\ldots=\sum c_{m-j},$$

and

$$\sum_{-\infty}^{\infty}\frac{c_m}{m+\mu+j}=\sum\frac{c_{m-1}}{m+\mu+j-1}=\ldots=\sum\frac{c_{m-j}}{m+\mu} \quad (51)$$

provided all these series converge. Equations (50) can be written as

$$(m-j+\mu-1)(p^2-1)c_{m-j}+(2a-2m+2j+3)kpc_{m-j-1}$$
$$-(\mu+2a-m+j+2)k^2c_{m-j-2}=0.$$

Dividing this by $\mu+m$ and adding for all values of m,

$$(p^2-1)\Sigma c_{m-j}-2kp\Sigma c_{m-j-1}+k^2\Sigma c_{m-j-2}-(j+1)(p^2-1)\Sigma_j$$
$$+(2\mu+2a+2j+3)kp\Sigma_{j+1}-(2\mu+2a+j+2)k^2\Sigma_{j+2}=0$$

which is equivalent to (45) if $k=p\pm 1$. This analysis holds even if Σc_m diverges, provided that $c_m\rightarrow 0$, and that all Σ_j converge.

It has thus been proved that for any values of a and p if the real part of p is not zero, and if the series

$$z^a e^{pz} \sum_{m=-\infty}^{\infty} c_m (kz)^{-m} \gamma(m+\mu, kz) \tag{52}$$

is convergent, this series is a solution of Bessel's differential equation in the form

$$\frac{d^2F}{dz^2} + \frac{1}{z}\frac{dF}{dz} - \left(1 + \frac{\nu^2}{z^2}\right)F = 0 \tag{53}$$

provided that $k = p+1$ if the real part of p is positive, and $k = p-1$ if the real part of p is negative; that $(\mu+a)^2 = \nu^2$; and that the coefficients c_m satisfy the recurrence relation (50).

5. *Convergence of the series.*

5.1. To discuss the convergence of this series, write (50) in the form

$$(m+\mu-1)[(p-1)c_m - pc_{m-1}] = (\mu+2a-m+2)[pc_{m-1} - (p+1)c_{m-2}]$$

with the real part of p positive, assuming $k = p+1$, from § 4.5. If $p \neq 1$, this can be written as

$$\frac{(m+\mu-1)c_m - (m-a-\frac{3}{2})c_{m-1}}{(m-a-\frac{3}{2})c_{m-1} - (m-\mu-2a-2)c_{m-2}} = \frac{p+1}{p-1}$$

and if c_m is bounded as $m \to \pm\infty$

$$\frac{(m+\mu-1)c_m - (m-a-\frac{3}{2})c_{m-1}}{mc_m - (m-\mu-a-\frac{1}{2})c_{m-1}} \to 1$$

so that

$$\frac{mc_m - (m-\mu-a-\frac{1}{2})c_{m-1}}{(m-1)c_{m-1} - (m-\mu-a-\frac{3}{2})c_{m-2}} \to \frac{p+1}{p-1}.$$

But $|(p+1)/(p-1)| > 1$, and hence, as $m \to \infty$,

$$|mc_m - (m-\mu-a-\tfrac{1}{2})c_{m-1}| \to \infty.$$

The series therefore does not converge if $p \neq 1$, unless (50) can be satisfied by a sequence of values for c_m in which $c_m = 0$, for all $m > m_0$.

Without loss of generality we can say that it does not converge if $p \neq 1$, unless $c_m = 0$ for all positive m, and that condition can be satisfied if and only if

$$\mu + 2a = 0, \quad a = -\nu. \tag{54}$$

5.2. But if $p = 1$, equation (50) can be written as

$$(2a-2m+3)c_{m-1} = 2(\mu+2a-m+2)c_{m-2}$$

$$\frac{c_m}{c_{m-1}}=2.\frac{\mu+2a-m+1}{2a-2m+1}=1+\frac{2\mu+2a+1}{2a-2m+1}$$

from which it is evident that $\sum_{-\infty}^{\infty} c_m$ converges as $m\to\pm\infty$ if the real part of $\mu+a>\frac{1}{2}$ while $\sum_{-\infty}^{\infty}\frac{c_m}{\mu+m}$ converges if the real part of $\mu+a>-\frac{1}{2}$, so that by **2**, the series determined by (52) with (55) converges if the real part of $\mu+a>-\frac{1}{2}$.

The series so determined is

$$c_0 z^a e^z \sum_{m=-\infty}^{\infty} (-2z)^{-m}\gamma(m+\mu,\ kz)\frac{\Gamma(\mu+2a+1)}{\Gamma(\mu+2a-m+1)}\cdot\frac{\Gamma(-a+\frac{1}{2})}{\Gamma(m-a+\frac{1}{2})}$$

or

$$z^a e^z \sum_{m=-\infty}^{\infty} (-2z)^{-m}\gamma(m+\mu,\ kz)\binom{\mu+a-\frac{1}{2}}{m-a-\frac{1}{2}}, \qquad (56)$$

where $\binom{n}{r}$ denotes $\frac{\Gamma(n+1)}{\Gamma(n-r+1)\,\Gamma(r+1)}$; if $c_0=\binom{\mu+a-\frac{1}{2}}{-a-\frac{1}{2}}$.

6. *Equating the series to a Bessel Function.*

6.1. It has been shown in the last section that

$$F(z)\equiv\frac{e^z(2z)^a}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\sum_{m=-\infty}^{\infty}\binom{\nu-\frac{1}{2}}{m-a-\frac{1}{2}}(-2z)^{-m}\gamma(\nu+m-a,\ 2z) \qquad (57)$$

is a convergent solution of Bessel's differential equation (53). From **2** it is evident that the principal part of this function in the neighbourhood of $z=0$ is

$$\frac{(2z)^\nu}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\sum_{m=-\infty}^{\infty}\binom{\nu-\frac{1}{2}}{m-a-\frac{1}{2}}\frac{(-1)^m}{\nu+m-a}, \qquad (58)$$

and therefore the function is $KI_\nu(z)$, where K is some constant.

6.2. To evaluate K we use the fact that

$$\frac{\frac{\partial}{\partial a}F(z)}{F(z)}=\frac{\frac{\partial K}{\partial a}}{K} \qquad (59)$$

and consider the behaviour of $\dfrac{\partial F}{\partial a}\Big/F$ as z tends to infinity.

$$\frac{\partial}{\partial a}F(z)=\frac{e^z(2z)^a}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\sum_{-\infty}^{\infty}\binom{\nu-\frac{1}{2}}{m-a-\frac{1}{2}}(-2z)^{-m}\gamma(\nu+m-a,\,2z)R_m,$$

where R_m denotes the ratio of any term in $\dfrac{\partial F}{\partial a}$ to the corresponding term in $F(z)$. It is

$$\log 2z+\frac{\Gamma'(m-a+\frac{1}{2})}{\Gamma(m-a+\frac{1}{2})}-\frac{\Gamma'(\nu-m+a+1)}{\Gamma(\nu-m+a+1)}-\frac{\gamma'(\nu+m-a,\,2z)}{\gamma(\nu+m-a,\,2z)},\quad(60)$$

which tends to zero as $m\rightarrow\pm\infty$ and is a bounded function of m.

Now, as z tends to infinity, the principal part of $F(z)$ consists of terms with large values of m. That is to say, given any positive number ε and any positive integer m_0, we can find a positive number x_0 such that for all values of z whose real part exceeds x_0,

$$\left|\frac{e^z(2z)^a}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\sum_{m=-\infty}^{m_0}\binom{\nu-\frac{1}{2}}{m-a-\frac{1}{2}}(-2z)^{-m}\gamma(\nu+m-a,\,2z)\right|<\varepsilon\,|F(z)|.$$

It follows thence that $\dfrac{\partial F}{\partial a}\Big/F$ can be made as small as we like by making z sufficiently large. But equation (59) shows that it is independent of z, and it is therefore zero. Hence K is independent of a. Now when $a=-\frac{1}{2}$, or half of any odd integer,

$$F(z)=\frac{e^z}{(2\pi z)^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\sum_{m=0}^{\infty}\binom{\nu-\frac{1}{2}}{m}(-2z)^{-m}\gamma(\nu+\tfrac{1}{2}+m,\,2z)$$
$$=I_\nu(z),$$

so that $K=1$ in that instance. It is therefore 1 for all values of a, and

$$I_\nu(z)=\frac{e^z(2z)^a}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\sum_{m=-\infty}^{\infty}\binom{\nu-\frac{1}{2}}{m-a-\frac{1}{2}}(-2z)^{-m}\gamma(\nu+m-a,\,2z)\quad(61)$$

7. An immediate corollary, from consideration of the principal part in the neighbourhood of $z=0$, is

$$\sum_{m=-\infty}^{\infty}\binom{\nu-\frac{1}{2}}{m-a-\frac{1}{2}}\frac{(-1)^m}{\nu+m-a}=\frac{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}{4^\nu\,\Gamma(\nu+1)}=B(\nu+\tfrac{1}{2},\,\nu+\tfrac{1}{2}),$$

which is a generalisation of the well-known relation

$$\sum_{m=0}^{\infty}\binom{\nu-\frac{1}{2}}{m}\frac{(-1)^m}{\nu+m+\frac{1}{2}}=B(\nu+\tfrac{1}{2},\nu+\tfrac{1}{2});$$

and of which another special case, obtained by putting $a=-\nu$, is

$$\sum_{m=0}^{\infty}\binom{\nu-\frac{1}{2}}{m}\frac{(-1)^m}{\nu-m+\frac{1}{2}}=B(\nu+\tfrac{1}{2},\nu+\tfrac{1}{2}).$$

The analogous generalisation of

$$\sum_{m=0}^{\infty}\binom{\nu-\frac{1}{2}}{m}(-1)^m=0 \quad\text{to}\quad \sum_{m=-\infty}^{\infty}\binom{\nu-\frac{1}{2}}{m-a}(-1)^m=0$$

follows from a result given by Whittaker and Watson*, viz.

$$B(p,q)+B(p+1,q)+B(p+2,q)+\ldots=B(p,q-1)$$

by making p tend to $-\infty$.

8. Lemma—*Evaluation of a determinant.*

8.1. In the proof of the uniqueness of the result of **4**, we need to consider the determinants formed by the coefficients in equations (45). These are special cases of the determinant whose general form is

$$\begin{vmatrix} \dfrac{1}{k} & \dfrac{1}{k+a_1} & \dfrac{1}{k+a_1+a_2} & \cdots\cdots & \dfrac{1}{k+\sum_1^n a_r} \\ \dfrac{1}{k+b_1} & \dfrac{1}{k+b_1+a_1} & \dfrac{1}{k+b_1+a_1+a_2} & \cdots\cdots & \dfrac{1}{k+b_1+\sum_1^n a_r} \\ \dfrac{1}{k+b_1+b_2} & \dfrac{1}{k+b_1+b_2+a_1} & \cdots & \cdots\cdots & \dfrac{1}{k+b_1+b_2+\sum_1^n a_r} \\ \vdots & \vdots & \vdots & \vdots\;\vdots & \vdots \\ \dfrac{1}{k+\sum_1^n b_r} & \dfrac{1}{k+\sum_1^n b_r+a_1} & \cdot & \cdot\;\cdot & \dfrac{1}{k+\sum_1^n b_r+\sum_1^n a_r} \end{vmatrix}$$

* *Modern Analysis* (1920), p. 261.

The lemma states that the value of this determinant* is

$$\frac{\prod_{r=1}^{n}\prod_{s=r}^{n}\left\{(a_r+a_{r+1}+\ldots+a_s)(b_r+b_{r+1}+\ldots+b_s)\right\}}{\prod_{r=1}^{n}\prod_{s=1}^{n}(k+a_1+a_2+\ldots+a_r+b_1+b_2+\ldots+b_s)}. \tag{62}$$

This can be proved as follows: When the determinant is expanded, and the terms put on a common denominator, the denominator will be the product of the denominators of all the elements in the determinant, and the numerator will be a homogeneous expression of degree $n(n+1)$ in $k, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$.

The determinant can be seen to vanish whenever any of the factors in the numerator of the expression (62) is equal to zero. There are $n(n+1)$ such factors, and since the expression giving the value of the determinant is of degree $n(n+1)$ in the numerator, there can only be an additional constant factor K. When the determinant is expanded, and the terms put on a common denominator, the term $b_1^n b_2^{n-1} \ldots b_n \; a_1^n a_2^{n-1} \ldots a_n$ occurs in the numerator only in the term corresponding to the product of the elements of the leading diagonal. By comparing this with the same term occurring in the expression (62), we see that K is 1.

The lemma is thus proved.

8.2. A corollary obtained by putting all the a's and b's equal to 1 is

$$\begin{vmatrix} \frac{1}{k} & \frac{1}{k+1} & \frac{1}{k+2} & \cdots & \frac{1}{k+n} \\ \frac{1}{k+1} & \frac{1}{k+2} & \frac{1}{k+3} & \cdots & \frac{1}{k+n+1} \\ \frac{1}{k+2} & \frac{1}{k+3} & \frac{1}{k+4} & \cdots & \frac{1}{k+n+2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{1}{k+n} & \frac{1}{k+n+1} & \frac{1}{k+n+2} & \cdots & \frac{1}{k+2n} \end{vmatrix} = [1\,!\,2\,!\,3\,!\ldots n\,!]^2\Pi,$$

* The evaluation of this determinant has been proposed as a problem in *The Mathematics Student*, VI, 2 (1938) p. 92, and I have to thank Mr. T. K. Raghavachari for suggesting a short proof.

where Π denotes the product of all the elements in the determinant. The other factor is:

$$[1!\,2!\,3!\ldots n!]^2 = [G(n+2)]^2,$$

the square of a Barnes's G-function.

The minor of the element in the $(r+1)$th row and the $(s+1)$th column in the last determinant can be found from the lemma, from a determinant of order n, putting

$$a_1 = a_2 = \ldots = a_{r-1} = 1,\ a_r = 2,\ a_{r+1} = \ldots = a_{n-1} = 1,$$
$$b_1 = b_2 = \ldots = b_{s-1} = 1,\ b_s = 2,\ b_{s-1} = \ldots = b_{n-1} = 1.$$

Let Δ_{rs} denote the required minor (without regard to sign), and let Π_{rs} denote the product of all its elements. Then

$$\frac{\Delta_{rs}}{\Pi_{rs}} = \binom{n}{r}\binom{n}{s}[(n-1)!\,(n-2)!\ldots 3!\,2!\,1!]^2$$

and

$$\frac{\Delta_{rs}}{\Delta} = \frac{1}{r!\,(n-r)!\,s!\,(n-s)!}\,\frac{\Pi_{rs}}{\Pi} = \frac{\prod_{i=0}^{n}(k+r+i)\prod_{i=0}^{n}(k+s+i)}{r!\,(n-r)!\,s!\,(n-s)!\,(k+r+s)}.$$

9. *Uniqueness of the solution.*

9.1. If there is more than one set of values of c_m satisfying the conditions for a solution, then by subtracting one such set from another we obtain a set of values of a_m such that

$$\sum_{m=-\infty}^{\infty} \frac{a_m}{\mu+m+i} = 0, \tag{63}$$

where i is any positive integer or zero. We shall show with certain assumptions as to the convergence of the series that no solution of the equations is possible except that in which $a_m = 0$ for all m.

Consider first the simple case in which the sequence a_m is from 0 to ∞ only, and not from $-\infty$ to $+\infty$, and suppose that Σa_m converges. Let $f(x)$ represent $\sum_0^\infty a_m x^m$, which is a continuous function of x, $0 \leqslant x \leqslant 1$. Equations (63) are then equivalent to

$$\int_0^1 x^{\mu+i-1} f(x)\,dx = 0,\ i \text{ being any positive integer or zero,}$$

from which it follows that $f(x)$ is identically zero, $0 \leqslant x \leqslant 1$, and therefore that $a_m = 0$ for all m.

9.2. Now the equations (63) can be written

$$\frac{a_{-l}}{\mu-l}+\frac{a_{-l+1}}{\mu-l+1}+\ldots+\frac{a_{-1}}{\mu-1}+\frac{a_0}{\mu}+\ldots+\frac{a_h}{\mu+h}=\varepsilon_0,$$

$$\frac{a_{-l}}{\mu-l+1}+\frac{a_{-l+1}}{\mu-l+2}+\ldots+\frac{a_{-1}}{\mu}+\frac{a_0}{\mu+1}+\ldots+\frac{a_h}{\mu+h+1}=\varepsilon_1, \quad (64)$$

. . . .

where ε_r denotes

$$-\sum_{i=-\infty}^{i=-l-1}\frac{a_i}{\mu+i+r}-\sum_{i=h+1}^{i=+\infty}\frac{a_i}{\mu+i+r}.$$

Taking $n+1$ of these equations, with $n=l+h$, eliminate all the a's except a_0, using the lemma of 8. We have

$$a_0=\frac{\mu(\mu+1)\ldots(\mu+n)}{l!\,h!}\sum_{r=0}^{n}\frac{(\mu-l+r)(\mu-l+r+1)\ldots(\mu+h+r)}{r!\,(n-r)!\,(\mu+r)}\varepsilon_r. \quad (65)$$

The argument of 9.1 shows that for any fixed l, if all the ε's tend to zero as h tends to infinity, then a_0 tends to zero. That is to say, if, given l and ε we can find h such that $\varepsilon_r < \varepsilon$ for all r and all $n > n_0$, then there exists a function $f(\varepsilon)$, such that $f(\varepsilon)$ tends to zero as ε tends to zero, and such that $|a_0| < f(\varepsilon)$. Now the expression for a_0 given by (65) is a function of which the modulus decreases as l increases if the real part of $\mu+r > 0$, as it is for all except a finite number of terms.

And instead of taking the first $n+1$ equations of (64), we can take those from the jth to the $(j+n)$th, where j is sufficiently large for the real part of $\mu+j$ to be positive. If μ is written for $\mu+j$, (65) will still represent the equations to be considered. Now we assume that $\sum\frac{a_m}{\mu+m+j}$ converges uniformly all j, as it will if $\sum\frac{a_m}{\mu+m}$ converges, and hence given any ε we can find l_0 and h_0 such that for all $l > l_0$ and $h > h_0$, all the ε_r's are less than ε. Then the argument above shows that for $l = l_0$ and $h > h_0$ the value given by (65) for a_0 will be less than $f(\varepsilon)$, and hence also for $l > l_0$ and $h > h_0$. But $f(\varepsilon)$ can be made as small as we like by making ε sufficiently small. And therefore $a_0 = 0$. The argument can of course be applied to a_r for any finite r. We see then that if all the series $\sum\frac{a_m}{\mu+m+j}$ converge as they must if a

series $\Sigma a_m w^m \gamma(\mu+m, w)$ is to converge, and if $\sum \frac{a_m}{\mu+m+j} = 0$ for all j, then all the a's are 0. It follows then that for given a and p there is not more than one sequence of values of c_m which will make

$$z^a e^{pz} \Sigma c_m (kz)^{-m} \gamma(\mu+m, kz)$$

a convergent series, and a solution of the Bessel equation of order ν, μ being determined by $\nu^2 = (\mu+a)^2$.

10. *The corresponding series with (complete) Gamma functions.*

The discussion of a series obtained by changing the incomplete Gamma functions in (57) or in (52) to the corresponding Gamma functions might be approached (since the Gamma functions are not functions of z) by considering the expansion in integral powers of z of

$$z^{-a} e^{-pz} I_\nu(z), \text{ and of } z^{-a} e^{-pz} I_{-\nu}(z).$$

It is useful however to follow the analysis of **4**: it can readily be shown that (57) remains a formal solution of the modified Bessel equation when the incomplete Gamma functions are respectively replaced by the corresponding Gamma functions, and more generally, that the condition which the coefficients c_m in (52) must satisfy in order that it should be a formal solution remains unaltered when Gamma functions are written for incomplete Gamma functions. For, if in equation (31) $\Gamma(m+\mu)$ is written for $\gamma(m+\mu, kz)$, then equation (33) becomes

$$z^a e^{pz} \Sigma c_m (kz)^{-m} \left[\frac{(a-m)^2 - \nu^2}{z^2} + \frac{2(a-m)+1}{z} p + p^2 - K^2 \right] \times \Gamma(m+\mu) = 0,$$

and equating coefficients of powers of z, we obtain for the constants c_m a recurrence relation which is exactly the same as (50), with ν^2, of course, equal to $(\mu+a)^2$.

We see therefore that the result of **4** can be stated:

If for any given values of a and p a sequence a_m is found, such that

$$z^a e^{pz} \sum_{m=-\infty}^{\infty} a_m z^m \tag{66}$$

is a formal solution of the modified Bessel differential equation, then

$$z^a e^{pz} \sum_{m=-\infty}^{\infty} a_m \frac{\gamma(\nu+m-a, kz)}{\Gamma(\nu+m-a)} z^m \tag{67}$$

will be a solution, whenever it is convergent, if $k = p \pm 1$.

The solution so obtained will be convergent if $p = \pm 1$ and $k = 2p$, if the real part of ν is greater than $-\frac{1}{2}$ for any value of a. The previously known case is $a = -\frac{1}{2}$ (or half of any odd integer). It is seen that there is a corresponding generalisation of the asymptotic expansion, viz.

$$\frac{e^z (2z)^a}{\pi^{\frac{1}{2}} \Gamma(\nu+\frac{1}{2})} \sum_{m=-\infty}^{\infty} \binom{\nu-\frac{1}{2}}{m-a-\frac{1}{2}} (-2z)^{-m} \Gamma(\nu+m-a),$$

where $\binom{\nu-\frac{1}{2}}{m-a-\frac{1}{2}}$ denotes $\dfrac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu-m+a+1)\,\Gamma(m-a+\frac{1}{2})}$.

The series (67) may converge for other values of p if $\nu+a$ is an integer, but since in that case the series (66) is a convergent series with positive powers of z, this is not of so much interest.

11. *Conclusion.*

The substance of this paper (except § 6.2) was contained in a thesis accepted by the University of Edinburgh in 1938. My thanks are due to Dr. A. C. Aitken, F. R. S., for suggesting methods for the investigation of a certain probability distribution, and for his stimulating direction, and interest in the work. The distribution function can, in certain special cases, be expressed in a form involving the Bessel function; other parts of the thesis, more specifically concerned with the probability distribution, will, I hope, be shortly published elsewhere.

PAIRS OF INVERSE MODULS

By F. W. LEVI, *University of Calcutta.*

[Received 29 September 1939].

Two submoduls A and A' of a (commutative) field F will be said to be *inverse*, if to every element $a \neq 0$ of A, there exists in A' the inverse element $a' = a^{-1}$, and conversely. If A' is the same modul as A, then A is said to be *self-inverse*. The rational homogeneous functions of x, y, z of order m with coefficients from an arbitrary field K, e.g. form a modul which is inverse to a modul formed by homogeneous functions of order $-m$. If $m \neq 0$, the two moduls have no common element, besides zero; if $m = 0$, the modul is a field and therefore self-inverse. An instance of a self-inverse modul which is not a field, is given by the purely imaginary numbers. In this paper, a complete survey of the pairs of inverse moduls will be given. It will be shown that the fields of characteristic 2 differ largely in this respect from the other fields. The main result of this paper is the following

THEOREM. *If the characteristic of F is different from 2, then one gets all the pairs of inverse submoduls which contain elements $\neq 0$ by*

$$A = aK$$
$$A' = a^{-1}K, \qquad (1)$$

where K is a subfield and a is an element of F. If in particular a is an element of K, then $A = A' = K$; if a is not an element of K, but a^2 is, then $A = A'$ is self-inverse, but not a field; if a^2 is not an element of K, then $A \neq A'$ and these inverse moduls have no element in common besides zero.

If the characteristic of F is equal to 2, then a similar theorem holds; in this case, K is not necessarily a subfield but a self-inverse modul containing 1. Unlike the case of characteristic $\neq 2$, these moduls need not be fields, and the inverse moduls may have common elements $\neq 0$, without being identical. This class of moduls needs a special discussion which of course will be given in this paper. In 5 the theory is applied to the Geometry of the Euclidean plane.

1. *Self-inverse moduls.*

Let A be a self-inverse submodul of a field F. If α is an element of F such that for every element a of A, αa is contained in A, then α is said to be a *multiplicator* of A. There exist at least two multiplicators, the zero element 0 and the unit element 1.

If α and β are multiplicators, and a is an element of A, then $(\alpha-\beta)a$ and $\alpha(\beta a) = (\alpha\beta)a$, are obviously elements of A, thus $\alpha-\beta$ and $\alpha\beta$ are multiplicators; hence the multiplicators form a ring. This holds for the multiplicators of every submodul of F. As A is supposed to be self-inverse, $1:\alpha(1:a) = a:\alpha$ is an element of A (provided $a \neq 0$). Hence we have

PROPOSITION 1. *The multiplicators of a self-inverse modul A form a field $M(A)$.*

The connection between A and $M(A)$ will be investigated now.

PROPOSITION 2. *If a is an element of a self-inverse modul A, then a^2 is an element of $M(A)$.*

PROOF. One has to prove that if a and b are elements of A, then a^2b is also an element of A, and without loss of generality we suppose that a and b are different from 0 and ab is different from 1. Then the following elements are also contained in A:

$$1:b, a-1:b = (ab-1):b, b:(ab-1)-1/a = 1:(a^2b-a), a^2b-a, a^2b.$$

From this proposition it follows in particular that all the odd powers of an element of A belong to A. The example of the purely imaginary numbers (which form a self-inverse modul) shows that the corresponding does not always hold for the even powers.

PROPOSITION 3. *If a and b are elements of a self-inverse modul A, then $2ab$ belongs to $M(A)$.*

From Prop. 2 it follows that $(a+b)^2$, a^2, b^2 belong to $M(A)$ and therefore also $(a+b)^2-a^2-b^2 = 2ab$ does.

In the case of characteristic 2, this proposition is trival, as $2ab = 0$. If however the characteristic is different from 2, this proposition has some consequences which do not hold for characteristic 2.

2. *Self-inverse moduls in fields of characteristic $\neq 2$.*

Let F be of a characteristic different from 2, and A be a self-inverse modul in F. $M(A)$ is a subfield of F and it contains

therefore an element 1:2. From Proposition 3 therefore it follows that $M(A)$ contains the products ab of every pair of elements a, b of A. Hence A contains every product abc if a, b, c are elements of A.

Suppose 1 is an element of A, then $ab1$ and $a1/b1$ are elements of A.

Hence:

PROPOSITION 4. *Let A be a self-inverse submodul of a field of a characteristic different from 2; if A contains 1, then A is a field.*

Suppose now that A contains a pair a, b of elements $\neq 0$, the product of which is also an element of A. Then it follows from (2) that $(ab)(1:a)(1:b) = 1$ is also an element of A, and therefore A is a field. Hence:

PROPOSITION 5. *Let A be a self-inverse submodul of a field of a characteristic different from 2; if A contains the product $ab \neq 0$ of one pair of elements of A, then A is a field.*

Obviously Prop. 4 is a special case of Prop. 5 as $1 = a(1:a)$. Consider now that $(c)(a)(1:b)$ is an element of A. Therefore every quotient $a:b$ of two elements of A belongs to $M(A)$. Hence the field $M(A)$ is identical with the quotients of the pairs of elements and the modul A can be represented by

$$A = a\,M(A), \tag{3}$$

where $a \neq 0$ is a fixed but arbitrary element of A. From Prop. 2 it is known that a^2 belongs to $M(A)$. If a itself is an element of $M(A)$, then $M(A)$ is identical with A; if it does not, no non-vanishing element of $M(A)$ belongs to A. Hence

PROPOSITION 6. *Let A be a self-inverse submodul of a field of a characteristic different from 2; then A can be represented by (3), where $a \neq 0$ denotes an arbitrary fixed element of A. The modul is identical with $M(A)$ if and only if a is an element of $M(A)$. At any rate a^2 is an element of $M(A)$.*

On the other hand it is obvious that a modul A represented by (3) is self-inverse if $M(A)$ is a field and contains a^2; in this case $M(A)$ is indeed composed of the multiplicators of A.

3. *Self inverse moduls of characteristic 2.*

The propositions of **2** have been proved under the supposition that F is of a characteristic different from 2. This restriction is necessary as will be shown by an example. Before that, some

interesting and nearly paradoxical formulae holding in fields of characteristic 2 will be mentioned:

$$a+b=a-b,$$
$$(a+b)^2=a^2+b^2$$
$$[\phi\,(x,y,\ldots,z)]^2=\phi_2\,(x^2,\,y^2,\ldots,z^2),$$

where the coefficients of ϕ_2 are the squares of the corresponding coefficients of ϕ.

Let now T be a field of characteristic 2, and x, y be two indefinites not belonging to T; it is supposed that no algebraic equation in x and y with coefficients from T holds. Consider the fields $F = T(x,y)$ and $M = T(x^2, y^2)$; then the square of every element of F belongs to M. Consider further the modul A formed by the elements

$$\alpha=ax+by+c,$$

where a, b, c belong to M. As α^2 belongs to M,

$$\alpha^{-1}=(\alpha^{-2}a)x+(\alpha^{-2}\,b)y+(\alpha^{-2}c)$$

is an element of A, provided $\alpha \neq 0$. The modul A is therefore self-inverse. As 1 is an element of A, only elements α of A can be multiplicators. $x\alpha$ belongs to A if and only if $b=0$, and $y\alpha$ does if and only if $a=0$. Hence α cannot be a multiplicator unless $a=b=0$. M is therefore equal to the field of the multiplicators of A. This modul A is a vector-space of rank 3 over the field M, whereas in the case of a characteristic different from 2, the self-inverse moduls are vector-fields of rank 1 over the field of the multiplicators. This statement means the same thing as Proposition 6.

The example just given can easily be generalised in the following manner. Let $x_1, x_2, \ldots$ be indefinites in any finite or infinite (may be non-enumerable) number $\nu > 1$ and let these elements be not connected by any algebraic equation with coefficients from T. Put

$$F' = T(x_1, x_2, \ldots)$$
$$M' = T(x_1^2, x_2^2, \ldots)$$

and let A' be the modul of all the elements. Let

$$\beta = b_0+b_1\,x_{i_1}+\ldots+b_m x_{i_m} \text{ for } m=0,\,1,\,\ldots$$

$i_1,\ldots, i_m$ being any collection of indices which may occur, and b_0, $b_1,\ldots, b_m$ being elements of M'

A' is a vector-space of rank $\nu+1$ over the field M', furthermore A' is not a ring, but a self-inverse modul, M' being its field of multiplicators. Every subspace A'' of A' is also a self-inverse modul, and $M(A'') = M'$.

To consider the general case of self-inverse submoduls of a field F of characteristic 2, suppose B to be such a modul, and its elements to be well ordered. Omit all those elements which are linearly dependent of elements of lower index with coefficients from $M(B)$. The remaining elements form a well ordered set

$$\beta_1, \beta_2, \ldots \tag{4}$$

of linearly independent elements. The vectorspace over $M(B)$ generated by (4) is composed of elements of B. On the other hand it is easy to prove that every element of B depends on (4). Let there be elements of B which are independent of (4), then there is one of them which has the smallest index in the original order of the elements, say β. As β is one of the previously omitted elements, it depends on elements of lower index, but as these elements depend on elements (4), the element β itself depends on them. Hence B is the vectorspace generated by (4), and it follows from the example given before that there is no upper limit for the power of the set (4). The elements of (4) may be connected by algebraic equations with coefficients out of $M(B)$. As the squares of the elements of B are elements of $M(B)$, the equations can be reduced to the form

$$\Sigma c_{i_1,\ldots,i_m} \beta_{i_1} \ldots \beta_{i_m} = 0, \tag{5}$$

i.e. to equations which are linear for every particular β_i which occurs. Hence (5) is equivalent to the representation of β_{i_m} as a rational function of $\beta_{i_1}, \ldots, \beta_{i_{m-1}}$, and as the squares are contained in $M(B)$, one can reduce these rational functions to integral functions of the type

$$\beta_{i_m} = \Sigma c_{i_1,\ldots,i_{m-1}} \beta_{i_1} \ldots, \beta_{i_{m-1}}. \tag{6}$$

Omit therefore in the well ordered set (4) those elements which can be represented as polynomials in the β_i of minor indices with coefficients out of $M(B)$; then the remaining elements of (4) form a well ordered set

$$\gamma_1, \gamma_2, \ldots \tag{7}$$

of elements which are connected by such algebraic equations only as follow from the fact that every γ_i is a square root of an element

of $M(B)$. The elements (4) can be represented as polynomials in the elements (7). Let

$$\beta_k = \Sigma c^k_{i_1,\ldots,i_n} \gamma_{i_1} \ldots \gamma_{i_n} \qquad (8)$$

The representation (8) is unique, otherwise γ_{i_n} could be represented as a polynomial in the other γ_i's.

Let K be the field $M(B)(\gamma_1, \gamma_2, \ldots)$, then K is a vector-space over $M(B)$ generated by the linearly independent elements

$$1, \gamma_1, \gamma_2, \ldots, \gamma_1\gamma_2, \gamma_1\gamma_3, \ldots, \gamma_1\gamma_2\gamma_3, \ldots, \gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_n}, \ldots \qquad (9)$$

B is a subspace of it which contains $\gamma_1, \gamma_2, \ldots$; on the other hand every subspace of this kind is a self-inverse modul for which the elements of $M(B)$ are multiplicators and which is not a field unless the modul is K itself. As the elements (9) are independent over $M(B)$, the squares of these elements are elements of $M(B)$ which are independent over the field consisting of the elements of $M(B)$; this condition is necessary and sufficient for the linear independence of the elements (9).

To get all the self-inverse moduls of characteristic 2 which are not fields, one starts from an arbitrary field M_1. Let M_2 be the field of the squares of the elements of M_1. Select a set $c_1, c_2, \ldots$ of elements of M such that there does not hold any linear relation between the terms

$$1, c_1, c_2, \ldots, c_1c_2, c_1c_3, \ldots, c_{i_1}c_{i_2}\ldots c_{i_n}, \ldots$$

with coefficients out of M_2. The square root of any one of these terms is not contained in M_1, otherwise the term itself is contained in M_2 and therefore dependent on 1.

Let γ_i be the square root of c_i (i taking all indices occurring), then

$$F = M_1(\gamma_1, \gamma_2, \ldots)$$

is a field which can be represented as a vector-space over M_1 with (9) as basis. Every subspace of it is a self-inverse modul. On the other hand every self-inverse modul of characteristic 2 can be represented as a vector-space of this kind, and as has been shown just before, one may suppose without any loss of generality that every γ_i is contained in it. Hence one gets all the self-inverse moduls of characteristic 2 which are not fields as the subfields B of all the vector-spaces F which are not F itself.

This representation is not unique. Every element of M_1 is a multiplicator of B; hence $M(B) \supset M_1$, but it is not necessary that M_1 is equal to $M(B)$.

Consider, e.g. $F = M_1(\gamma_1, \gamma_1, \gamma_3, \gamma_4)$, and let B be the vector-space over M generated by $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_1\gamma_2\gamma_4, \gamma_1\gamma_3\gamma_4$. Then $\gamma_1\gamma_4$ is a multiplicator of B, and B is a vector-space generated by $\gamma_1, \gamma_2, \gamma_3$ over $M_1(\gamma_1\gamma_4) = M(B)$. If in general B is of a finite rank γ over M_1, the field of the multiplicators is an extension of M_1 of an order 2^k, and therefore B is of rank $r2^{-k}$ over $M(B)$. If therefore B is of an odd rank over M_1, then $M_1 = M(B)$. This condition is sufficient, but not necessary.

If B contains any one of the multiplicators, it contains 1 and therefore the whole field of the multiplicators. In this case either B is a field itself, and therefore $B = M(B)$, or B contains an element β which is not contained in $M(B)$; then the vector-space generated by 1 and β is a field. Hence a self-inverse modul which is not a field, but which contains its multiplicators, has submoduls with a larger field of multiplicators.

Examples for self-inverse moduls B which are vector-spaces over $M(B)$ of any rank $\nu+1>2$ have been established before (ν may be of finite or infinite power). Examples for rank 1 are obvious. It will be shown now that the rank cannot be equal to two.

Applying the general method of construction of all moduls B, one can suppose without loss of generality that every γ_i is contained in B. Hence there are not more than two elements γ_i. Thus M is a vector-space over M_1 either generated by 1, γ or generated by γ_1, γ_2. In the first case B is a field, and $B = M(B) = M_1 (\gamma)$. In the second case $\gamma_1\gamma_2$ is a multiplicator, $M(B) = M_1(\gamma_1\gamma_2)$ and $B = \gamma_1 M(B)$ is of rank 1 over $M(B)$.

That Prop. 5 cannot be extended to the case of characteristic 2 follows from the fact that B may contain $M(B)$ without being identical with it. It may be mentioned also that such self-inverse moduls B which do not contain any non-vanishing multiplicator may contain particular pairs of elements which can be multiplied inside B. Consider e.g. the modul

$$a\gamma + b\delta + c\gamma\delta, \tag{10}$$

where $a, b, c, \gamma^2, \delta^2$ belong to a field M^1 of characteristic 2, γ and δ being fixed. If on the other hand a self-inverse modul contains two non-multiplicators γ, δ and their product, it contains a subspace of elements of rank 3. These spaces have the property that to every particular element (10) there corresponds a "plane"

of those elements which are "multipliable" with it. Indeed the product of

$$x\gamma+y\delta+z\gamma\delta$$

and (10) belongs to that 3-dimensional space if and only if the coordinates x, y, z satisfy

$$x(a\gamma^2)+y(b\delta^2)+z(c\gamma^2\delta^2)=0,$$

which is a linear homogeneous equation with coefficients out of M_1.

4. *Pairs of moduls.*

Let A and A' be inverse submoduls of an aibitrary field throughout this section.

PROPOSITION 7. *If A is a ring and contains elements $\neq 0$, then A is a field and identical with A'.*

PROOF. One needs only to prove that if A contains $a\neq 0$, it contains also $1:a$. If A contains a then A' contains $1/a+1/a^2=(a+1):a^2$. Hence A contains $a-a^2:(a+1)=a:(a+1)$; hence A' contains $(a+1):(a-1):a=1$. Hence A contains 1 and therefore $a-1$, and A' contains $1+1:(a-1)=a:(a-1)$. Hence A contains $1-(a-1):a=1:a$. Hence the theorem.

The *multiplicators* of A are those elements of F which multiplied with every element of A result in a product contained in A.

The multiplicators of A form obviously a ring which is inverse to the ring of the multiplicators of A'. Further 1 is a multiplicator of A. Hence from Proposition 7 we get

PROPOSITION 8. *The multiplicators of A form a field which is simultaneously the field of the multiplicators of A'. This field will be denoted by $M(A)=M(A')$.*

Suppose that A contains the product $ab\neq 0$ of two of its elements a and b. Then A' contains the product $a^{-1}\,b^{-1}$ of two of its elements a^{-1}, b^{-1}. If therefore it can be proved in consequence of the supposition made, that a polynomial $\phi(ab)$ with coefficients from $M(A)=M(A')$ must be an element of A, then it follows that $\phi(a^{-1},b^{-1})$ is an element of A' and hence $1:\phi(a^{-1},b^{-1})$ is an element of A. This consideration will abbreviate the proof of the following proposition.

PROPOSITION 9. *If a, b, $ab\neq 0$ are elements of A, then $a:b$ and $b:a$ belong to A.*

PROOF. As $a+ab$ belongs to A, it follows from the consideration just before that $1:[a^{-1}+(ab)^{-1}] = ab:(b+1)$ belongs to A. Hence $a-ab:(b+1) = a:(b+1)$ and therefore $\frac{b^{-1}+1}{1:a} = \frac{a}{b}+a$ belong to A. Hence $a:b$ is an element of A and the same holds for $b:a$.

Let

$$A_o = A \cap A' \tag{11}$$

be the meet of two inverse pairs. Then the following proposition holds.

PROPOSITION 10. *A_0 is a self-inverse modul. If in particular a, b and $ab \neq 0$ are elements of A, then a, b, ab, $a:b$, $b:a$ are elements of A_0.*

PROOF. The meet of two submoduls of a field is a modul, which may consist of 0 only. If $a \neq 0$ is an element of A_0, it is an element of A and of A'; hence $1:a$ is an element of A' and of A and therefore of A_0. Hence A_0 is a self-inverse modul, which possibly consists of 0 only.

If a, b and $ab \neq 0$ are elements of A, then it follows from Prop. 9 that $a:b$ and $b:a$ belong to A; as this pair of elements is inverse, it belongs also to A' and therefore to A_0. As A_0 is a self-inverse modul, $a^2:b^2$ and $b^2:a^2$ belong to $M(A)$. Now a, $b:a$ and $a(b:a) = b$ belong to A. Hence $a:(b:a) = a^2:b$ is an element of A_o and as $b^2:a^2$ is a multiplicator of A_0, b is an element of A_0. By symmetry it follows that a is an element of A_0. As furthermore $b:a$ is an element of A_o, and a^2 is a multiplicator of A_0, ab is an element of A_0.

PROPOSITION 11. *If A contains an element $b \neq 0$ of $M(A)$, then $A = A'$ is self-inverse.*

PROOF. If $a \neq 0$ is an element of A, then a, b, and $ab \neq 0$ belong to A; hence it follows from Proposition 10 that a belongs to A_0. Hence every element of A belongs to A_0. Hence the proposition.

PROPOSITION 12. *If A and A' are inverse moduls containing elements different from 0, then*

$$A = a\,S$$
$$A' = a^{-1}S, \tag{12}$$

where S is a self-inverse modul containing 1.

PROOF. If σ is any element of F, then σA and $\sigma^{-1}A'$ are inverse moduls. Let a be an element of A, then $a^{-1}A = S$ contains 1 and is therefore self-inverse. Hence $S = aA'$. Hence the proposition.

To prove the theorem stated at the beginning of this paper, one has only to apply Proposition 4. If the characteristic of F is $\neq 2$, then S is equal to a field K, and from (3) it follows that $K = M(A)$. If a is an element of K, then $A = K = A'$.

If $a^2 = k$ is an element of K, and therefore $a^{-1} = a:k$, then $A' = akK = aK = A$. If on the other hand A and A' have a common element $b \neq 0$, then

$$b = ak_1 = a^{-1}k_2$$

and therefore $$a^2 = k_1k_2 = k$$

is an element of K, as K is a field. Hence the meet A_0 consists of the zero element only unless $A = A' = A_0$ is self-inverse. Thus the theorem is proved.

If the characteristic of F is equal to 2, then S may not be a field. Further A and A' may have common elements without being identical as will be shown by the following example.

Let T be a field of characteristic 2, α and β be transcendentals over T not connected by any algebraic equation with coefficients out of T. Put

$$T(\alpha^4, \beta^4) = M,$$

and let S be the vector-space generated by $\alpha^2, \beta^2, 1$. Then S is a self-inverse modul, $M(S) = M$.

$A = \alpha\beta S$ is inverse to $A' = \alpha^3\beta^3 S$. These moduls are vector-spaces over M.

A	is generated by		$\alpha^3\beta,\ \alpha\beta^3,\ \alpha\beta$
A'	,,	,,	$\alpha^3\beta,\ \alpha\beta^3,\ \alpha^3\beta^3$
$A_0 = A \cap A'$		,,	$\alpha^3\beta,\ \alpha\beta^3$.

5. *Application of the theory to Plane Geometry.*

Let L be a point set in the plane such that if A, B, C are points of L, and the vectors AB and CD are equal, then D belongs to L. Every set having this property is said to be a *lattice*. A lattice is therefore invariant for every parallel displacement defined

by two of the lattice points. A modul of complex numbers represented in the usual manner by points of the plane forms a lattice containing the point zero. On the other hand, if one selects any point of a lattice as the point 0 of the complex plane, the lattice points form a modul of complex numbers.

Let O, E, A be points of a plane, $O \neq E$, $O \neq A$; let A' be the point which is symmetric to A with respect to the straight line OE, and let A_{OE} be the point symmetric to A' with respect to the circle with centre O and radius OE. If O is the zero point, and E is the unit point of the complex plane, then A and A_{OE} are represented by inverse complex numbers. On the other hand the transformation $A \rightarrow A_{OE}$ can be represented by the transformation $\alpha \rightarrow \alpha^{-1}$ of the complex numbers when O and E are chosen as the points zero and one.

Let now L_1 and L_2 be lattices, L_1 is supposed to contain O, E and the points B_{OE} for which $B \neq O$ is a point of L_2; L_2 is supposed to contain the points A_{OE}, for which $A \neq 0$ is a point of L_1. If O and E are chosen as the zero and the unit point, then L_1 is represented by a modul containing every integral number.

Thus L_2 is represented by a set of numbers which contains e.g. $\frac{1}{2}$, $\frac{1}{4}$ and therefore $0 = \frac{1}{4} - (\frac{1}{2} - \frac{1}{4})$. Hence L_2 is represented by a modul, and this modul is inverse to the modul representing L_1 and containing 1. From Prop. 11 it follows that the two moduls are identical and self-inverse. Hence $L_1 = L_2$. Consider now the lattices which can be represented by self-inverse moduls containing 1.

THEOREM. *Let L be a lattice containing O, E and all the points A_{OE} corresponding to the lattice points $A \neq O$. Then L has the following properties:*

(1) *If A_1, A_2, A_3 is similar to B_1, B_2, B_3 in the strict sense (the orientation of both the triangles* being the same) and if five of these points are lattice points the sixth point too is a point of L.*

(2) *If $A \neq B \neq C$ are lattice points, then C_{BA} is.*

PROOF. Represent $O, E, A_1, A_2, A_3, B_1, B_2, B_3, A, B, C, C_{BA}$ by $0, 1, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \alpha, \beta, \gamma, \gamma'$.

* The triangles may be degenerate.

From Proposition 4 it follows that L is represented by a field of complex numbers. Now

$$\frac{\beta_1-\beta_3}{\beta_1-\beta_2}=\frac{\alpha_1-\alpha_3}{\alpha_1-\alpha_2}$$

and

$$(\gamma'-\beta)(\gamma-\beta)=(\alpha-\beta)^2.$$

Hence β_3 is a rational function of $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$ and γ' is a rational function of α, β, γ. Hence the theorem.

ON SOME SERIES AND INTEGRALS INVOLVING k_n FUNCTIONS

By N. G. SHABDE, *College of Science, Nagpur.*

[Received 21 June, 1939].

The object of this note is to obtain certain series and integrals involving k_n-functions. These formulae do not seem to have been noted before.

1. Starting with the generating function of* Laguerre polynomials, namely

$$\frac{1}{(1-t)^{\alpha+1}} \exp\left(\frac{xt}{t-1}\right) = \sum_{n=0}^{\infty} t^n L_n^{(\alpha)}(x) \tag{1.1}$$

and using the relation

$$L_n^{(1)}(x) = (-1)^n (n+1) \frac{e^{x/2}}{x} k_{2n+2}(x/2) \tag{1.2}$$

we obtain

$$\frac{1}{(1-t)^2} \exp\left(\frac{2xt}{t-1}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n t^n (n+1) e^x k_{2n+2}(x)}{2x} \tag{1.3}$$

or

$$\frac{2xe^{-x}}{(1+t)^2} \exp\left(\frac{2xt}{1+t}\right) = \sum_{n=0}^{\infty} (n+1) t^n k_{2n+2}(x). \tag{1.4}$$

Changing t into it in (1·4) and separating real and imaginary parts we obtain

$$\sum_{r=0}^{\infty} (2r+1)(-1)^r t^{2r} k_{4r+2}(x)$$

$$= \frac{2xe^{-x}}{(1+t^2)^2} \exp\left\{\frac{2xt^2}{1+t^2}\right\} \left[(1-t^2)\cos\left(\frac{2xt}{1+t^2}\right) + 2t \sin\left(\frac{2xt}{1+t^2}\right)\right] \tag{1.5}$$

*H. Bateman, "The k-function, a particular case of the confluent hypergeometric function," *Trans. Am. Math. Soc.* 33, 817-831. See page 819, formula (2.5).

and

$$\sum_{r=0}^{\infty}(2r+2)(-1)^r t^{2r+1} k_{4r+4}(x)$$

$$=\frac{2xe^{-x}}{(1+t^2)^2}\exp\left\{\frac{2xt^2}{1+t^2}\right\}\left[(1-t^2)\sin\left(\frac{2xt}{1+t^2}\right)-2t\cos\left(\frac{2xt}{1+t^2}\right)\right]. \quad (1.6)$$

2. Another generating function* for Laguerre polynomials is given by

$$\frac{e^{-t}J_\alpha(2\sqrt{xt})}{(xt)^{\alpha/2}}=\sum_{n=0}^{\infty}\frac{t^n L_n^{(\alpha)}(x)}{\Gamma(\alpha+n+1)}. \quad (2.1)$$

Making use of (1.2) we get

$$\frac{2xe^{-(t+x)}}{(2xt)^{1/2}}J_1(2\sqrt{2xt})=\sum_{n=0}^{\infty}\frac{(-1)^n t^n k_{2n+2}(x)}{n!}. \quad (2.2)$$

Change t into it in (2.2), and for the separation of $J_1(2\sqrt{i}\sqrt{2xt})$ into real and imaginary parts, make use of a result† by Savidge. We obtain

$$i\left\{\frac{2xe^{-x}}{(2xt)^{1/2}}\right\}(\cos t+i\sin t)\left\{\operatorname{ber}'(z)-i\operatorname{bei}'(z)\right\}_{z=2\sqrt{2xt}}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n i^n t^n k_{2n+2}(x)}{n!}. \quad (2.3)$$

Equating the real and imaginary parts in (2.3), we have

$$\sum_{r=0}^{\infty}\frac{(-1)^r t^{2r}}{(2r)!}k_{4r+2}(x)=$$

$$\sqrt{\frac{2x}{t}}\,e^{-x}\left\{-\sin t\operatorname{ber}'(z)+\cos t\operatorname{bei}'(z)\right\}_{z=2\sqrt{2xt}} \quad (2.4)$$

and

$$\sum_{r=0}^{\infty}\frac{(-1)^{r+1}t^{2r+1}k_{4r+4}(x)}{(2r+1)!}=$$

$$\sqrt{\frac{2x}{t}}\,e^{-x}\left\{\cos t\operatorname{ber}'(z)+\sin t\operatorname{bei}'(z)\right\}_{z=2\sqrt{2xt}}. \quad (2.5)$$

* G.N. Watson, "A note on the polynomials of Hermite and Laguerre," *Journ. Lond. Math. Soc.* 13 (1938), 204-9. See p. 207.

† "Tables of the ber and bei, ker and kei functions with further formulae for their computation", *Phil. Mag.* XIX (1910), p. 49.

3. From a theorem* given by Erdélyi, it is easy to deduce that

$$\sum_{m=0}^{n} c_m\Gamma(\mu+m+3/2)p^{-m}(-1)^{\mu+m-\frac{1}{2}}k_{2\mu+2m+1}(x/2p)$$

$$= p^{\mu+\frac{1}{2}}x^{\frac{1}{2}}e^{\frac{1}{2}(x/p)}\int_0^\infty e^{-py}y^{\mu}P(y)J_1(2\sqrt{xy})\,dy, \qquad (3.1)$$

where $R(p) > 0$; $R(\mu+\frac{1}{2}) > -1$ and $P(y) = \sum_{m=0}^{n} c_m y^m$.

Now use for $P(y)$ the Jacobi polynomial given by

$$P_n^{(\alpha,\beta)}(1-2y) = \frac{(-1)^n\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}\sum_{r=0}^{n}\frac{(-n)_r(n+\alpha+\beta+1)_r}{r!\,(\alpha+1)_r}y^r. \qquad (3.2)$$

This gives us

$$\int_0^\infty e^{-py}y^{\mu}J_1(2\sqrt{xy})P_n^{(\alpha,\beta)}(1-2y)\,dy =$$

$$p^{-\mu-\frac{1}{2}}x^{-\frac{1}{2}}e^{-\frac{1}{2}(x/p)}\frac{(-1)^n\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}$$

$$\times\sum_{r=0}^{n}\frac{(-n)_r(n+\alpha+\beta+1)_r}{r!\,(\alpha+1)_r}\Gamma(\mu+r+3/2)p^{-r}(-1)^{\mu+r-\frac{1}{2}}k_{2\mu+2r+1}(x/2p),$$

where $R(p) > 0$, $R(\mu+\frac{1}{2}) > -1$, $\alpha > -1$, $\beta > -1$. (3.3)

Taking $\mu = m+\frac{1}{2}$, where m is zero or an integer greater than -2, we obtain

$$\int_0^\infty e^{-py}y^{m+\frac{1}{2}}J_1(2\sqrt{xy})P_n^{(\alpha,\beta)}(1-2y)\,dy$$

$$= p^{-m-1}x^{-\frac{1}{2}}e^{-\frac{1}{2}(x/p)}\frac{(-1)^n\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}$$

$$\times\sum_{r=0}^{n}\frac{(-n)_r(n+\alpha+\beta+1)_r}{r!\,(\alpha+1)_r}\Gamma(m+r+2)p^{-r}(-1)^{m+r}k_{2m+2r+2}(x/2p). \qquad (3.4)$$

*"On certain Hankel Transforms", *Quart. Journ. Math.* (Oxford Series) 9 (1938) 196 203. See Theorem 5.

Similarly, formula (8) of Erdélyi's paper* gives for the k_n-function the result

$$\int_0^\infty \frac{e^{-y}J_1(2\sqrt{xy})k_{2n+2}\left(\frac{y}{2p}\right)e^{y/2p}dy}{\sqrt{y}}$$

$$=\frac{1}{p}\left(1-\frac{1}{p}\right)^n\frac{1}{x^{\frac{1}{2}}}(1-p)e^{x\left(\frac{2p-1}{2(1-p)}\right)}k_{2n+2}\left(\frac{x}{2-2p}\right). \quad (3.5)$$

4. An integral involving two Bessel functions and two k_n-functions can be derived from an integral given by Erdélyi†. This integral by Erdélyi is of a general type containing a number of $M_{k,m}$-functions. Taking note of the results

$$J_n(z)=\frac{z^{-\frac{1}{2}}M_{0,n}(2iz)}{2^{2n+\frac{1}{2}}i^{n+\frac{1}{2}}\Gamma(n+1)} \quad (4.1)$$

and

$$(-1)^{n-1}k_{2n}(z/2)=M_{n,\frac{1}{2}}(z), \quad (4.2)$$

our required integral comes out as

$$\int_0^\infty e^{-pz}z^{q+1}k_{2n_1}\left(\frac{\alpha_1 z}{2}\right)k_{2n_2}\left(\frac{\alpha_2 z}{2}\right)J_{n_3}(\alpha_3 z)J_{n_4}(\alpha_4 z)dz=$$

$$\frac{(-1)^{n_1+n_2}\alpha_1\alpha_2(2i\alpha_3)^{n_3+\frac{1}{2}}(2i\alpha_4)^{n_4+\frac{1}{2}}}{\Gamma(n_3+1)\,\Gamma(n_4+1)\sqrt{\alpha_3\alpha_4}\,2^{2n_3+2n_4+1}\,i^{n_3+n_4+1}}$$

$$\times\left\{p+\tfrac{1}{2}(\alpha_1+\alpha_2+2i\alpha_3+2i\alpha_4)\right\}^{-q-n_3-n_4-4}\Gamma(q+n_3+n_4+4)$$

$$\times F_A\Big[q+n_3+n_4+4;\,1-n_1,\,1-n_2,\,\tfrac{1}{2}+n_3,\,\tfrac{1}{2}+n_4;\,2,\,2,\,1,\,1;$$

$$\frac{\alpha_1}{p+\frac{1}{2}(\alpha_1+\alpha_2+2i\alpha_3+2i\alpha_4)},\,\frac{\alpha_2}{p+\frac{1}{2}(\alpha_1+\alpha_2+2i\alpha_3+2i\alpha_4)},$$

$$\frac{2i\alpha_3}{p+\frac{1}{2}(\alpha_1+\alpha_2+2i\alpha_3+2i\alpha_4)},\,\frac{2i\alpha_4}{p+\frac{1}{2}(\alpha_1+\alpha_2+2i\alpha_3+2i\alpha_4)}\Big],$$

where $R(q+n_3+n_4+4)>0$ and F_A is the Lauricella hypergeometric function of four variables. (4.3)

From (1.2) and a result by Watson‡, we note the integral

* *loc. cit.*

† "Über einige bestimmte integrale in denen die Whittakerschen $M_{k,m}$-functionen auftreten", *Math. Zeit*, 40 (1936), 693-702. See p. 696.

‡ "Another note on Laguerre polynomials", *Jour. Lond. Math. Soc.* 14 (1939) 19-22. See p. 21.

$$\int_0^\pi \exp(-\sqrt{xy}\cos\theta)\, J_{\frac{1}{2}}(\sqrt{xy}\sin\theta) \times \frac{e^{\frac{x+y}{2}+\sqrt{xy}\cos\theta}\, k_{2n+2}\left(\frac{x+y}{2}+\sqrt{xy}\cos\theta\right)\sin\theta\, d\theta}{\left\{\frac{x+y}{2}+\sqrt{xy}\cos\theta\right\}\left\{\frac{1}{2}\sqrt{xy}\sin\theta\right\}^{\frac{1}{2}}}$$

$$= (-1)^n \frac{\pi^{\frac{1}{2}}}{xy}\, k_{2n+2}(x/2)\, k_{2n+2}(y/2).$$

Similarly, another integral* yields for k_n-function the result

$$\int_0^\infty e^{-py} y^{n+\frac{1}{2}} J_1(2\sqrt{xy})\, dy$$

$$= (-1)^n \frac{\Gamma(n+2)}{p^{n+1}} x^{-\frac{1}{2}} e^{-\frac{1}{2}(x/p)} k_{2n+2}(x/2p)\,;\ R(p) > 0. \qquad (4.5)$$

An integral representation for the product of two k_n-functions similar to (4.4) can be obtained from a result by Howell†. The result is

$$\int_0^{2\pi} e^{-i\gamma + x\sqrt{ab}\, e^{i\gamma}} L_n^0\left\{x(a+b-2\sqrt{ab}\cos\gamma)\right\} d\gamma$$

$$= 2\pi(n+1)\, \frac{e^{\frac{a+b}{2}x}}{\sqrt{ab}\, x}\, k_{2n+2}\left(\frac{ax}{2}\right) k_{2n+2}\left(\frac{bx}{2}\right) \qquad (4.6)$$

* Watson, *Theory of Bessel functions*, Art. 153.

† "Integral representation for products of Weber's Parabolic cylinder functions", *Phil. Mag.* 1938, p. 456.

ON CERTAIN FUNCTIONAL EQUATIONS

By V. GANAPATHY IYER, *Annamalai University.*

[Received 18 April 1939]

1. The equation

$$P^2+Q^2=1 \tag{1}$$

can be satisfied by means of the integral functions

$$P=\cos(az+b), \qquad Q=\sin(az+b). \tag{2}$$

The question may be asked whether there are other solutions of the equation (1). In this paper I propose to find the most general solution in *integral functions* of the equation (1), and to discuss certain allied questions.

2. THEOREM 1. *Let $f(z)$ be any integral function. Then the functions*

$$P=\cos[f(z)], \qquad Q=\sin[f(z)], \tag{3}$$

satisfy the equation (1). *Conversely, if P and Q are integral functions satisfying* (1), *then we can find an integral function $f(z)$ such that the relations* (3) *hold.*

PROOF. The first part of the theorem is evident. To prove the second part, suppose P and Q are integral functions satisfying (1). Since

$$P^2+Q^2\equiv(P+iQ)(P-iQ)=1, \tag{4}$$

it follows that $P+iQ$ is an integral function having no zeros. Therefore we can find an integral function $f(z)$ such that

$$P+iQ=e^{if(z)}. \tag{5}$$

From (4) and (5) we get

$$P-iQ=e^{-if(z)}. \tag{6}$$

The relations (3) follow from (5) and (6)*.

2.1. THEOREM 2. *If the solutions* (3) *of* (1) *are periodic then $f'(z)$ is periodic. Conversely, if $g(z)$ be an integral function such that $g'(z)$ is periodic, we can choose the constant μ such that $f(z)=\mu g(z)$ put in* (3) *gives a periodic solution of* (1).

* The apparently more general equation $aP^2+bQ^2=1$ can be reduced to (1) by putting $U=a^{\frac{1}{2}}P$, $V=b^{\frac{1}{2}}Q$ so that we get $U^2+V^2=1$.

PROOF. Suppose the solutions (3) are periodic, and let λ be a period. Then

$$\cos[f(z+\lambda)]-\cos f(z) = 0,$$

from which we deduce that for each given z there must be an *integer* κ (positive, negative or zero) such that one of the following relations

$$\left.\begin{aligned} &\text{(i)}\ f(z+\lambda)-f(z) = 2\kappa\pi \\ &\text{(ii)}\ f(z+\lambda)+f(z) = (2\kappa+1)\pi \end{aligned}\right\} \qquad (7)$$

hold. Suppose (i) of (7) holds for $z = z_0$ and for a certain κ. Then since κ is an integer, (i) of (7) must hold *in the neighbourhood of z_0 for the same* κ. Since $f(z+\lambda)-f(z)$ is an integral function, it follows that (i) *of* (7) *holds for that particular* κ *for all* z. Hence differentiating, we get that $f'(z)$ is periodic with period λ. If (ii) of (7) holds for $z = z_0$ and for some κ, we can repeat the above argument and deduce the relation $f'(z+\lambda)+f'(z) = 0$, from which we get $f'(z+2\lambda) = f'(z)$, so that $f'(z)$ is periodic with period 2λ. Conversely, if $g'(z)$ is periodic with period λ we get

$$g(z+\lambda)-g(z) = c, \text{ a constant.} \qquad (8)$$

If $c = 0$, we take $f(z) = g(z)$, $(\mu = 1)$. If $c \neq 0$, we choose μ so that $\mu c = 2\kappa\pi$ (κ an integer $\neq 0$) and take $f(z) = \mu g(z)$. Then from the identity

$$\cos\theta-\cos\phi = 2\sin\frac{\theta+\phi}{2}\sin\frac{\phi-\theta}{2}$$

we easily deduce that P in (3) is periodic. Similarly Q is periodic.

2.2. THEOREM 3. *If the solutions* (3) *are of finite order, then $f(z)$ in* (3) *is a polynomial so that P and Q are of integral order. If, in addition, the solution is periodic, then $P = \cos(az+b)$, $Q = \sin(az+b)$ so that these are the only periodic solutions of finite order of the equation* (1).

PROOF. If P and Q are of finite order, then $P+iQ$ in (5) is of finite order so that $f(z)$ is a polynomial. If, in addition, the solutions (3) are periodic, then, by Theorem 2, $f'(z)$ is periodic. But a polynomial is periodic only if it is a constant. Hence $f'(z) = a$, a constant. Therefore $f(z) = az+b$. This completes the proof of Theorem 3.

3. We next prove the following negative result.

THEOREM 4. *Let n be a positive integer $\geqslant 3$. Then there exist no integral functions P and Q satisfying the relation*

$$P^n + Q^n = 1, \tag{9}$$

except in the trivial case when P and Q are constants.

PROOF. Let $\omega = e^{\frac{\pi i}{n}}$ be an nth root of -1, and let $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ be the nth roots of unity. Then

$$P^n + Q^n \equiv (P-\omega Q)(P-\omega\alpha Q)(P-\omega\alpha^2 Q)\ldots(P-\omega\alpha^{n-1}Q) = 1. \tag{10}$$

If P and Q are integral functions satisfying (9) we deduce from (10) that the n integral functions

$$P-\omega Q,\ P-\omega\alpha Q, \ldots, P-\omega\alpha^{n-1}Q$$

are functions without zeros. It follows that the meromorphic function P/Q does not take the n values $\omega, \omega\alpha, \ldots, \omega\alpha^{n-1}$. But, by Picard's theorem, a meromorphic function can fail to take at most two values, unless it reduces to a constant. Hence we arrive at a contradiction when $n \geqslant 3$, unless P/Q is a constant. If P/Q is a constant, then P and Q are each of them constants in virtue of (9). So the theorem is proved.

3.1. From Theorem 3 we deduce that if $f(z)$ be an integral function, there cannot exist two different numbers a and b such that $f(z)-a$ and $f(z)-b$ are both perfect nth powers for $n \geqslant 3$. For, if $f(z)-a=P^n$, $f(z)-b=Q^n$, we get

$$\left[\frac{P}{(b-a)^{\frac{1}{n}}}\right]^n + \left[\frac{e^{\pi i/n}Q}{(b-a)^{\frac{1}{n}}}\right]^n = 1,$$

which cannot be satisfied when P and Q are integral functions. When $n=2$, two such constants might exist as is illustrated by the functions $\cos z+1$ and $\cos z-1$. But more than two constants cannot exist when $n=2$*.

3.2. Finally we may mention that the method of **2** can be used to find the general solution of the equation

$$aP^2 + 2bPQ + cQ^2 = e^{g(z)}. \tag{11}$$

where $g(z)$ is an integral function. In particular, the general solutions of

$$P^2 + Q^2 = e^{g(z)}, \tag{12}$$

* For proof of this and more precise results of this nature, see R. Nevanlinna, "*Functions Meromorphes*".

in integral functions are

$$P=e^{\frac{g(z)}{2}}\cos[f(z)],\ Q=e^{\frac{g(z)}{2}}\sin[f(z)]. \tag{13}$$

Also, if the polynomial

$$a_0z^n+a_1z^{n-1}+\ldots+a_n,\ n\geqslant 3 \tag{14}$$

has at least three distinct roots, the equation

$$a_0P^n+a_1P^{n-1}Q+a_2P^{n-2}Q^2+\ldots+a_nQ^n=e^{g(z)}, \tag{15}$$

where $g(z)$ is an integral function cannot be satisfied by integral functions P and Q. The proof of this is similar to that of Theorem 3.

AN INEQUALITY FOR THE ARITHMETICAL FUNCTION $g(x)$*

By S. M. SHAH, *Muslim University, Aligarh.*

[Received 17 July 1938]

Let $n = a_1 + a_2 + \ldots + a_\rho$, and $f(n)$ the maximum of the least common multiple of $a_1, a_2, \ldots, a_\rho$ for all such positive a's. Landau† has proved that if $\log f(x) = g(x)$, then

$$\lim_{x\to\infty} \frac{g(x)}{(x \log x)^{\frac{1}{2}}} = 1. \tag{1}$$

Let p denote a prime number. We prove here the following

THEOREM.

$$I(y) \leqslant g(x) \leqslant I(y) + O[(x \log x)^{\frac{1}{4}}],$$

where y *is the greatest integer such that* $\sum\limits_{p\leqslant y} p \leqslant x$

and $$I(y) = \sum_{p\leqslant y} \log p.$$

We deduce that

$$g(x) = (x \log x)^{\frac{1}{2}} + \tfrac{1}{2}\frac{x^{\frac{1}{2}} \log\log x}{(\log x)^{\frac{1}{2}}} + O\left(\sqrt{\frac{x}{\log x}}\right). \tag{2}$$

LEMMA 1. $$g(x) \geqslant I(y).$$

This is obvious since $e^{g(x)} = f(x) \geqslant \prod\limits_{p\leqslant y} p$.

LEMMA 2. *Let* $a = 0$ *or* 1, *and* α *be an absolute constant.*

Let $$z = (x \log x)^{\frac{1}{2}}\left\{1 + \frac{\log\log x - \alpha}{2 \log x}\right\} + a.$$

Then $$\sum = \frac{z^2}{2 \log z} + \frac{z^2}{4 \log^2 z} + \frac{z^2}{4 \log^3 z} + O\left(\frac{z^2}{\log^4 z}\right)$$

$$= x\left[1 + \frac{1}{\log x}\left(\tfrac{1}{4} - \alpha\right) + O\left\{\frac{(\log\log x)^2}{(\log x)^2}\right\}\right].$$

LEMMA 3. $$y = (x \log x)^{\frac{1}{2}}\left\{1 + \frac{\log\log x}{2 \log x} + O\left(\frac{1}{\log x}\right)\right\}.$$

* Abstract read at the Conference of the Indian Mathematical Society, Lucknow (March 1938).

† *Primzahlen.* Vol. I, 222-229.

For,
$$\sum_{p\leqslant z} p = -\sum_{n=1}^{z} \pi(n) + \pi(z)\{[z]+1\}$$

and
$$\sum_{n=1}^{z} \pi(n) = \sum_{n=2}^{z}\left\{\frac{n}{\log n}+\frac{n}{\log^2 n}+\frac{2n}{\log^3 n}+O\left(\frac{n}{\log^4 n}\right)\right\}$$
$$\doteq \int_2^z u\left\{\frac{1}{\log u}+\frac{1}{\log^2 u}+\frac{2}{\log^3 u}\right\}du+O\left(\int_2^z \frac{u\,du}{\log^4 u}\right)$$
$$=\frac{z^2}{2\log z}+\frac{3}{4}\frac{z^2}{\log^2 z}+\frac{7}{4}\frac{z^2}{\log^3 z}+O\left(\frac{z^2}{\log^4 z}\right)$$

Hence
$$\sum_{p\leqslant z} p=\frac{z^2}{2\log z}+\frac{z^2}{4\log^2 z}+\frac{z^2}{4\log^3 z}+O\left(\frac{z^2}{\log^4 z}\right). \tag{3}$$

Hence from Lemma 2 we get the result.

LEMMA 4. *Let p_r be the rth prime and k be such that*
$$p_2+p_3+\ldots+p_{k-1}<x\leqslant p_2+p_3+\ldots+p_k.$$
Let $q_1, q_2, \ldots, q_l$ be distinct primes and
$$q_1+q_2+\ldots+q_l\leqslant x.$$
Then $p_1p_2\ldots p_k>q_1q_2\ldots q_l.$ (4)

PROOF. From the p's and q's we delete all those primes that occur among both sides of (4). We remove $p_1=2$ from the left side of (4) whether or not it is a q. Let the remaining p's be
$$x_1, x_2, \ldots, x_m.$$

If there be no remaining p, then each p_r $(2\leqslant r\leqslant k)$ is a q_s $(1\leqslant s\leqslant l)$, and since
$$p_2+\ldots+p_k\geqslant x$$
$$q_1+\ldots+q_l\leqslant x$$
and so $p_2p_3\ldots p_k=q_1q_2\ldots q_l$, and the conclusion of the lemma is true. Let the remaining q's be $t_1, t_2, \ldots, t_n$. If there be no remaining q, then there is nothing to prove. We have
$$3\leqslant x_1<x_2<\ldots<x_m<t_1<t_2\ldots<t_n \tag{5}$$
$$x_1+x_2+\ldots+x_m\geqslant t_1+t_2+\ldots+t_n. \tag{6}$$
Now $x/\log x$ is a strictly increasing function of x for $x>e$, and so if $e<a<b$ then
$$b<a^{b/a};\ a>b^{a/b}. \tag{7}$$
Hence from (5) and (7),
$$x_1x_2\ldots x_m>t_1^{(x_1+\ldots+x_m)/t_1}$$
$$>t_1^{(t_1+\ldots+t_n)/t_1}$$
$$>t_1t_2\ldots t_n,$$

and (4) follows.

Now let $E = E(x)$ denote the set* of p's for which

$$\Sigma p^{\nu} \leqslant x. \tag{8}$$

and

$$\Pi p^{\nu} = e^{g(x)} \tag{9}$$

Let P be the largest prime whose square occurs as a term in the sum Σp^{ν} of (8)

If now $P < p \leqslant P^2 - P$, then p belongs to E; for otherwise we can increase the value of Πp^{ν} by substituting Pp instead of P^2, and this would not affect (8) since $P + p < P^2$.

Hence

$$g(x) > I(P^2 - P) - I(P).$$

Hence from (1), $c(x \log x)^{\frac{1}{2}} > P^2 + O\left(\frac{P^2}{\log P}\right)$

i.e.

$$P < A x^{\frac{1}{4}} (\log x)^{\frac{1}{4}}. \tag{10}$$

We can deduce (10) directly also. For

$$P^2 + \sum_{P < p < P^2 - P} p \leqslant x.$$

If we now use (3) we get (10).

Similarly if P_1 be the largest prime whose cube occurs in E then for $x > x_0$

$$P_1 < A x^{\frac{1}{6} + \varepsilon},$$

and so on.

Let y_1 be the smallest prime such that $\sum_{p \leqslant y} p \geqslant x + 2$.

Then

$$\begin{aligned} g(x) &\leqslant \sum_{p \leqslant y} \log p + 2 \sum_{p \leqslant P} \log p + \left\{ 3 \sum_{p \leqslant P} \log p + \ldots \right\}. \\ &\leqslant \sum_{p \leqslant y} \log p + O(\log x) + 2 \sum_{p \leqslant P} \log p + \left\{ \ldots \right\} \\ &\leqslant I(y) + O\left\{ x^{\frac{1}{4}} (\log x)^{\frac{1}{4}} \right\} + O\left(x^{\frac{1}{6} + 2\varepsilon}\right) \\ &= I(y) + O\left\{ (x \log x)^{\frac{1}{4}} \right\}. \end{aligned} \tag{11}$$

Now

$$I(y) = (x \log x)^{\frac{1}{2}} \left\{ 1 + \frac{\log \log x}{2 \log x} + O\left(\frac{1}{\log x}\right) \right\}. \tag{12}$$

Hence from (11), (12) and Lemma 1, we get (2).

Finally I must thank Dr. T. Vijayaraghavan for suggesting the problem to me and for helpful criticisms.

* *loc. cit.* p. 224.

THE SUMMATION OF THE INTEGRAL CONJUGATE TO THE FOURIER INTEGRAL OF THE FINITE TYPE AND OF THE CONJUGATE SERIES.

By RAMANATH MOHANTY, *Ravenshaw College, Cuttack.*

[Received 27th July, 1939].

1. *Introduction.* The integral conjugate to the Denjoy-Fourier Integral of the finite type [7]* is

$$\int_0^\infty (b_s \cos sx - a_s \sin sx)\, ds, \tag{1}$$

where

$$a_s = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos st\, dt,\ b_s = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin st\, dt, \tag{2}$$

and $f(x)$ is supposed to be integrable-D in $(-\pi, \pi)$ and to be zero outside this interval.

1.1. To consider the Cesaro summability of (1), we have to examine the behaviour of

$$J_0^{(k)}(n) = \int_0^n \left(1-\frac{s}{n}\right)^k (b_s \cos sx - a_s \sin sx)\, ds, \tag{3}$$

as $n \to \infty$ and for a given value of x. It has been proved [1] that if $k > -1$,

$$J_0^{(k)}(n) = \frac{n}{\pi}\int_{-\pi-x}^{\pi-x} f(x+t) g_k(nt)\, dt, \tag{4}$$

where

$$g_k(t) = \int_0^1 (1-u)^k \sin tu\, du. \tag{5}$$

Remembering the definition of $f(x)$, we can also write

$$J_0^{(k)}(n) = \frac{n}{\pi}\int_0^\infty \psi(t) g_k(nt)\, dt, \tag{6}$$

* The numbers in [ ] denote the references at the end.

where

$$\psi(t) = f(x+t) - f(x-t). \tag{7}$$

In (6) we can replace ∞ by $X = \max\,(\pi+x, \pi-x)$.

1.2. In **2**, I consider the summability (C, k) of (1) for positive integral values of k. In 2.2, I discuss the corresponding results for the allied Fourier series of $f(x)$. Results similar to these latter have been obtained by B. N. Prasad [8], but the methods employed and the results contained in this paper seem to be definitely simpler. In **3**, I resume the discussion of the integral (1) and find sufficient conditions for summability (C, k)—k positive integral—which are, in some respects, more general than those obtained by Miss Grimshaw [1]. In **4**, I obtain similar results for the conjugate Fourier Series. I prove a theorem more general than one of B. N. Prasad [8] by using Rieszian means in the place of Cesaro's means.

1.3. I take this opportunity to express my thanks to Prof. D. N. Sen for his kind interest and encouragement in the preparation of this paper. I also wish to express my thanks to the referee who has taken much trouble in improving the paper and putting it into a form convenient for publication.

2. THEOREM 1. *Let*

$$\omega_1(t) = \frac{1}{t}\int_0^{(t)} \psi(t)\,dt, \tag{8}$$

and let

$$\int_0^{\infty} \frac{\omega_1(t)}{t}\,dt \tag{9}$$

exist. Then the integral (1) *converges and has the value*

$$\frac{1}{\pi}\left[\omega_1(X) + \int_0^{\infty} \frac{\omega_1(t)}{t}\,dt\right]$$

provided

$$\lim_{n\to\infty}\int_p^{\infty} \frac{\psi(t)}{t}\cos nt\,dt = 0, \tag{10}$$

where $p = \frac{2m+1}{n}\,\frac{\pi}{2}$ and $2m+1 < n$.

PROOF. Here $X = \max\,(\pi+x, \pi-x)$. We have, from (5) and (6)

$$J_0^{(0)}(n) = \frac{1}{\pi}\int_0^{\infty} \psi(t)\frac{1-\cos nt}{t}\,dt, \tag{11}$$

since

$$g_0(nt) = \frac{1-\cos nt}{t}$$

We write (11) in the form

$$J_0^{(0)}(n) = \frac{1}{\pi}\int_p^{\infty}\frac{\psi(t)}{t}\,dt + \frac{1}{\pi}\int_0^p \psi(t)\,\frac{1-\cos nt}{t}\,dt - \frac{1}{\pi}\int_p^{\infty}\frac{\psi(t)}{t}\cos nt\,dt$$

$$= I_1 + I_2 - I_3, \text{ say.} \tag{12}$$

We have,

$$I_1 = \frac{1}{\pi}\,\omega_1(X) - \frac{1}{\pi}\omega_1(p) + \frac{1}{\pi}\int_p^{\infty}\frac{\omega_1(t)}{t}\,dt$$

$$I_2 = \frac{1}{\pi}\omega_1(p) - \frac{1}{\pi}\int_0^p t\omega_1(t)\,\frac{d}{dt}\left(\frac{1-\cos\ nt}{t}\right)dt,$$

on integration by parts and using the fact that $p = \dfrac{2m+1}{n}\,\dfrac{\pi}{2}$.

Now,

$$\frac{1}{\pi}\int_0^p t\omega_1(t)\,\frac{d}{dt}\left(\frac{1-\cos nt}{t}\right)dt$$

$$= -\frac{1}{\pi}\int_0^p\frac{\omega_1(t)}{t}dt + \frac{1}{\pi}\int_0^p\frac{\omega_1(t)}{t}\cos nt\ dt + \frac{1}{\pi}\int_0^p\frac{\omega_1(t)}{t}nt\sin nt\,dt$$

$$= -J_1 + J_2 + J_3, \text{ say.}$$

Using the fact that $\cos nt$ is of bounded variation in $(0,\ p)$, writing it as the difference of two increasing functions and applying the second mean-value theorem to each of the integrals thus obtained, we have

$$J_2 = O\left(\int_{\frac{2m+1}{n'}\frac{\pi}{2}}^{\frac{2m+1}{n}\frac{\pi}{2}}\frac{\omega_1(t)}{t}dt\right)$$

$$= o(1),$$

since (9) exists as a Cauchy integral at $t=0$. Similarly, since the total variation of $nt\sin nt$ in $(0,\ p)$ cannot exceed $np\int_0^p n|\cos nt|\ dt \leqslant n^2p^2 \leqslant \left(\frac{2m+1}{2}\pi\right)^2$, we can again use the second mean-value theorem and thus obtain the result

$$J_3 = o(1).$$

Combining these results, we get from (12)

$$J_0^{(0)}(n) = \frac{1}{\pi}\omega_1(X) + \frac{1}{\pi}\int_0^{\infty}\frac{\omega_1(t)}{t}dt - \frac{1}{\pi}\int_p^{\infty}\frac{\psi(t)}{t}\cos nt\ dt + o(1)$$

as $n \to \infty$. Theorem 1 follows from the last relation. Using the Riemann-Lebesgue theorem, it can be shown that (10) holds if

$$\lim_{n=\infty} \int_0^{\delta} \psi(t) \frac{\cos nt}{t} dt = 0 \tag{13}$$

and that (13) holds if

$$\int_0^t |d[t\psi(t)]| = O(t). \tag{14}$$

2.1. THEOREM 2. *Let*

$$\left.\begin{aligned}
\omega_1(t) &= \frac{1}{t}\int_0^t \psi(t)dt, \; J_0^0(n) = \frac{1}{\pi}\int_0^{\infty} \psi(t) \frac{1-\cos nt}{t} dt; \\
\omega_2(t) &= \frac{1}{t}\int_0^t \omega_1(t)dt, \; J_1^{(0)}(n) = \frac{1}{\pi}\int_0^{\infty} \omega_1(t) \frac{1-\cos nt}{t} dt; \\
&\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
\omega_m(t) &= \frac{1}{t}\int_0^t \omega_{m-1}(t)dt, \; J_m^{(0)}(n) = \frac{1}{\pi}\int_0^{\infty} \omega_m(t) \frac{1-\cos nt}{t} dt.
\end{aligned}\right\} \tag{15}$$

Let the integral

$$\frac{1}{\pi}\int_0^{\infty} \frac{\omega_{k+1}(t)}{t} dt \tag{16}$$

exist. Then (1) *is summable* (C, k) *to the sum*

$$\frac{1}{\pi}\left[\int_0^{\infty} \frac{\omega_{k+1}(t)}{t} dt + \sum_{m=1}^{k+1} \omega_m(X)\right] \tag{17}$$

provided

$$\lim_{n\to\infty} \int_0^{\delta} \omega_k(t) \frac{\cos nt}{t} dt = 0. \tag{18}$$

PROOF. We have

$$\begin{aligned}
J_0^{(1)}(n) &= \frac{1}{n\pi}\int_0^{\infty} \frac{\psi(t)}{t}\left[n - \frac{\sin nt}{t}\right] dt \\
&= \frac{\omega_1(X)}{\pi} + \frac{2}{n\pi}\int_0^{\infty} \omega_1(t)\frac{1}{t}\left(n - \frac{\sin nt}{t}\right) dt \\
&\quad - \frac{1}{\pi}\int_0^{\infty} \omega_1(t)\frac{1-\cos nt}{t} dt + o(1)
\end{aligned}$$

as $n \to \infty$, so that we get,

$$J_0^{(1)}(n) = \frac{1}{\pi}\omega_1(X) + 2J_1^{(1)}(n) - J_1^{(0)}(n) + o(1).$$

Similarly, integrating by parts, we get

$$\left.\begin{aligned}
J_0^{(k)}(n) &= \frac{1}{\pi}\omega_1(X)+(k+1)J_1^{(k)}(n)-kJ_1^{(k-1)}(n)+o(1)\\
J_1^{(k-1)}(n) &= \frac{1}{\pi}\omega_2(X)+kJ_2^{(k-1)}(n)-(k-1)J_2^{(k-2)}(n)+o(1)\\
&\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
J_{k-1}^{(1)}(n) &= \frac{1}{\pi}\omega_k(X)+2J_k^{(1)}(n)-J_k^{(0)}(n)+o(1),
\end{aligned}\right\} \tag{19}$$

where

$$J_k^{(m)}(n) = \frac{n}{\pi}\int_0^\infty \omega_k(t)g_m(nt)\,dt. \tag{20}$$

Using the fact that, if an integral is summable (C, k) it is summable (C, r) for $r > k$ and that, by Theorem 1,

$$J_k^{(0)}(n) = \frac{1}{\pi}\int_0^\infty \omega_k(t)\frac{1-\cos nt}{t}dt$$

has a limit, as $n\to\infty$, under the conditions of Theorem 2, we find that

$$\lim_{n\to\infty} J_k^0(n) = \lim_{n\to\infty} J_k^{(1)}(n),$$

and so from (19), $\lim\limits_{n\to\infty} J_{k-1}^{(1)}(n)$ exists. Again

$$\lim_{n\to\infty} J_{k-1}^{(1)}(n) = \lim_{n\to\infty} J_{k-1}^{(2)}(n),$$

and therefore $\lim\limits_{n\to\infty} J_{k-2}^{(2)}(n)$ exists by (19). Repeating this argument we find that $\lim\limits_{n\to\infty} J_0^{(k)}(n)$ exists, and from (19) we get

$$\begin{aligned}
\lim_{n\to\infty} J_0^{(k)}(n) &= \frac{1}{\pi}\sum_{m=1}^{k}\omega_m(X)+\lim_{n\to\infty} J_k^{(0)}(n)\\
&= \frac{1}{\pi}\sum_{m=1}^{k}\omega_m(X)+\frac{1}{\pi}\left[\omega_{k+1}(X)+\int_0^\infty \frac{\omega_{k+1}(t)}{t}dt\right]\\
&= \frac{1}{\pi}\left[\int_0^\infty \frac{\omega_{k+1}(t)}{t}dt+\sum_{m=1}^{k+1}\omega_m(X)\right].
\end{aligned}$$

This proves the theorem. The condition (18) could also be replaced by

$$\int_0^t |dt\omega_k(t)| = O(t). \tag{21}$$

2.2. In this section we shall consider similar results for the conjugate of the Fourier series of $f(x)$ integrable L in $(-\pi, \pi)$ and periodic with period 2π. When $k > 0$, the kth Riesz's mean (which is equivalent to kth Cesaro's mean) of the series [2] is

$$\sum_{m \leqslant n} (b_m \cos mx - a_m \sin mx) \left(1 - \frac{m}{n}\right)^k = \frac{n}{\pi} \int_0^\infty \psi(t) g_k(nt) dt. \quad (22)$$

The behaviour of (21) as $n \to \infty$ can be discussed in the same manner as (6). When $k = 0$, (21) does not hold but in this case we can write

$$\bar{S}_n^{(0)} = \sum_{m=1}^{n} (b_m \cos mx - a_m \sin mx)$$

$$= \frac{1}{2\pi} \int_0^\pi \psi(t) \cot \frac{t}{2} (1 - \cos nt) dt + \frac{1}{2\pi} \int_0^\pi \psi(t) \sin nt \, dt. \quad (23)$$

The second term in (22) is $o(1)$ as $n \to \infty$, and since $\psi(t)$ is odd and periodic we get

$$\bar{S}_n^{(0)} = \frac{1}{\pi} \int_0^\infty \psi(t) \frac{1 - \cos nt}{t} dt + o(1). \quad (24)$$

We can deal with (24) in the same manner as (22) and (6). Thus we get the following

THEOREM 3. *If the integral*

$$\frac{1}{\pi} \int_0^\infty \frac{\omega_{k+1}(t)}{t} dt$$

exists, then the conjugate series is summable (C, k) *and has the sum*

$$\frac{1}{\pi} \int_0^\infty \frac{\omega_{k+1}(t)}{t} dt,$$

provided that

$$\int_0^\delta \left| \frac{\omega_k(t)}{t} \right| dt$$

exists or that $\omega_{k+1}(t)$ *is of bounded variation or*

$$\int_0^t |\omega_{k+1}(t)| dt = O(t).$$

3. Resuming the discussion of the integral (1), let

$$\left.\begin{aligned}\Psi(t) &= \frac{1}{\pi}\int_t^\infty \frac{\psi(u)}{u}du - s,\ 0 < t < X;\\ &= 0,\ \text{for other values of } t\end{aligned}\right\}$$

and

$$\Psi_1(t) = \int_0^t \Psi(u)du,\ \Psi_r(t) = \int_0^t \Psi_{r-1}(u)du,\ r \geqslant 2. \tag{25}$$

It is known [1] that a sufficient condition for summability (C, k) of (1) to s is

$$\frac{\Psi_n(t)}{t^n} \to 0$$

as $t \to 0$, n being the integer next below $k-1$, or more generally

$$\frac{1}{t}\int_0^t \frac{|\Psi_{r-1}(u)|}{u^{r-1}}du \to 0$$

as $t \to 0$, where r is any positive integer less than k. Now the condition

$$\psi_{k-1}(t) = \frac{\Psi_{k-1}(t)}{t^{k-1}} \to 0 \text{ as } t \to 0 \tag{26}$$

is less restrictive than the two previous conditions and we propose to find what additional condition along with (26) will ensure summability (C, k) of (1). Now, if k is a positive integer, the condition for summability (C, k) of (1) is

$$\lim_{\eta\to 0}\ \overline{\lim_{n\to\infty}} \left| n^k \int_0^\eta \Psi_{k-1}(u)\gamma_k^{(k)}(nt)dt \right| = 0, \tag{27}$$

where $\gamma_k^{(k)}(nt)$ is the kth derivative of $\gamma_k(nt)$ with respect to t and

$$\gamma_k(t) = \int_0^1 (1-u)^{k-1}\cos tu\, du.$$

Splitting up the integral in (27) into two parts, from 0 to $1/n$ and $1/n$ to η, using (26) and the fact that

$$\gamma_k^{(k)}(nt) = \frac{A\cos nt}{(nt)^k} + O\left[\frac{1}{(nt)^{k+1}}\right]$$

as $t \to 0$, we find after some simplification that (27) reduces when (26) holds, to

$$\lim_{\eta\to 0}\ \overline{\lim_{n\to\infty}} \left| \int_{\frac{1}{n}}^{\eta} \psi_{k-1}(t)\frac{\cos nt}{t}dt \right| = 0. \tag{28}$$

But (28) is the necessary and sufficient condition that the series conjugate to $\psi_{k-1}(t)$ should be convergent to 0 at $t=0$. So this is the additional condition required.

3.1. When $k \geqslant 2$, a sufficient condition for summability (C, k) of (1) is

$$\psi_{k-1}(t) = o(1) \text{ and } \frac{1}{t}\int_0^t \frac{|\Psi_{k-2}(t)|}{t^{k-2}}dt = O(1), \tag{29}$$

because it can be shown that when (29) holds, the condition

$$\int_0^t |d[t\psi_{k-1}(t)]| = O(t)$$

is satisfied, and this latter is Young's test for the convergence of the conjugate series of $\psi_{k-1}(t)$ at $t=0$.

3.2. A necessary and sufficient condition for summability (C, k) of (1) to s is

$$\frac{1}{\pi}\int_0^\infty \psi(t)\frac{1-k\gamma_k(nt)}{t}dt \to s, \ (k > -1). \tag{30}$$

When $k>0$, $\frac{1-\gamma_k(nt)}{t}$ is of bounded variation in $(0, \infty)$. In this case (30) reduces to

$$\frac{1}{\pi}\int_0^\infty \Psi(t)\left[\frac{1-k\gamma_k(nt)}{t^2} + \frac{kn\gamma'_k(nt)}{t}\right]dt \to s + \frac{1}{\pi}\frac{\Psi(X)}{X}, \tag{31}$$

where $\Psi(t) = \int_0^t \psi(t)dt$. Let $a = s + \frac{1}{\pi}\frac{\Psi(X)}{X}$ If

$$\bar{\phi}(t) = \frac{1}{\pi}\int_t^\infty \frac{\Psi(u)}{u}du$$

and $\chi_0(t) = \bar{\phi}(t) - a$, the condition (31) on further integration by parts becomes

$$\lim_{n\to\infty}\lim_{\eta\to 0} n^2\int_\eta^\infty \chi_0(t)t\gamma_k''(nt)dt = 0. \tag{32}$$

3.3. Similarly using the functions

$$\chi_1(t) = \int_0^t \chi_0(u)du, \ \chi_r(t) = \int_0^t \chi_{r-1}(u)du \tag{33}$$

we can obtain the following conditions for summability:—

(i) $\lim_{\eta\to 0} \overline{\lim_{n\to\infty}} \left| n^2\int_0^\eta \chi_r(t)\frac{d^r}{dt^r}[t\gamma_k''(nt)]\right| = 0;$

(ii) $\frac{\chi_n(t)}{t^n} \to 0$ where n is the integer next below $k-2$;

(iii) $\frac{1}{t}\int_0^t \frac{|\chi_{r-1}(u)|}{u^{r-1}} du \to 0$ as $t \to 0$, where r is any integer less than $k-1$. The condition (31), and (i), (ii) and (iii) above are, in some respects, more general than those obtained by Miss Grimshaw [1]. Some further conditions can be derived from those stated above similar to (28).

4. We now proceed to discuss results similar to those of § 3-3.3 for the conjugate series of the Fourier Series of $f(x)$ which is integrable$-L$ in $(-\pi, \pi)$ and is periodic with period 2π. We have

$$S_{0,n}^{(k)} = \sum_{1 \leqslant m \leqslant n} \left(1-\frac{m}{n}\right)^k (b_m \cos mx - a_m \sin mx)$$

$$= \frac{1}{\pi}\int_0^\infty \psi(t) \frac{1-k\gamma_k(nt)}{t} dt, \ (k > 0). \tag{34}$$

If $k > 0$, $\frac{1-k\gamma_k(nt)}{t}$ is of bounded variation in $(0, \infty)$ and so integration by parts can be used in (34). Using this fact and proceeding as in **3** and keeping to the same notation, we can write the condition of summability (C, k) of (34) to s in one of the following forms:

(i) $\frac{1}{\pi}\int_0^\infty \Psi(t)\left[\frac{1-k\gamma_k(nt)}{t^2} + \frac{kn\gamma'_k(nt)}{t}\right]dt \to s$ as $n \to \infty$;

(ii) $\lim_{\eta\to\infty} kn^2\int_\eta^\infty \bar{\phi}(t)t\gamma_k''(nt)dt \to s$ as $n \to \infty$,

where $$\bar{\phi}(t) = \frac{1}{\pi}\int_t^\infty \frac{\Psi(t)}{t^2}dt;$$

(iii) If $\chi_1(t)$ exists, the necessary and sufficient condition for summability (C, k) to s can be written as

$$\lim_{n\to\infty} kn^2\int_0^\infty \bar{\phi}(t)t\gamma_k''(nt) \to s$$

and if $k > 2$, ∞ in the last integral can be replaced by any $\xi > 0$.

4.1. We conclude this paper with the statement and proof of the following

THEOREM 4. *At all points x where the integral*

$$\frac{1}{\pi}\int_0^\infty \frac{\Psi(t)}{t^2}dt \tag{35}$$

exists at least as a non-absolutely convergent integral, the conjugate series is summable (C, k) *to the sum* (34) *when* $k > 1$, *provided*

$$\theta(t) = \frac{1}{t}\int_0^t \frac{\Psi(t)}{t}dt = o(1) \text{ and } \bar{\theta}(t) = \frac{1}{t}\int_0^t \left|\frac{\Psi(t)}{t}\right| dt = O(1).$$

PROOF. By using a condition similar to (iii) of § 3.3, we can derive that if

$$\frac{1}{\pi}\int_0^\infty \frac{\Psi(t)}{t^2}dt \tag{36}$$

is summable (C, n), then the conjugate series is summable (C, k) for $k > n+2$. In particular, if (36) is convergent, the conjugate series is summable (C, k) for $k>2$. What we have to show is that if the additional conditions of Theorem 4 hold, the conjugate series is summable (C, k) for $k>1$. Let k and l be any two numbers greater than one. We have

$$S^{(k)}_{0,n} - S^{(l)}_{0,n} = \frac{n}{\pi}\int_0^\infty \Psi(t)\left[\frac{k\gamma'_k(nt)}{t} - \frac{l\gamma'_l(nt)}{t}\right]dt$$
$$-\frac{1}{\pi}\int_0^\infty \Psi(t)\left[\frac{k\gamma_k(nt)}{t^2} - \frac{l\gamma_l(nt)}{t^2}\right]dt. \tag{37}$$

If $\xi>0$ and $n>1/\xi$, we have

$$\left|\int_\xi^\infty \Psi(t)\frac{\gamma_k(nt)}{t^2}dt\right| = O\left(\frac{1}{n^\alpha}\int_\xi^\infty \frac{dt}{t^{\alpha+2}}\right) = o(1) \text{ as } n\to\infty,$$

where $\alpha=\min(k, 2)>1$. Also

$$\left|\int_\xi^\infty \Psi(t)\frac{nt\gamma'_k(nt)}{t^2}dt\right| = O\left(\frac{1}{n^{\beta-1}}\int_\xi^\infty \frac{dt}{t^{\beta+1}}\right) = o(1) \text{ as } n\to\infty,$$

where $\beta=\min(k, 3)>1$. Similarly the portion of the integrals in (37) between $(0, \lambda/n)$ can also be shown to be $o(1)$ as $n\to\infty$ in virtue of the hypothesis on $\theta(t)$ and $\bar{\theta}(t)$. Hence we get

$$\varlimsup_{n\to\infty} |S^{(k)}_{0,n} - S^{(l)}_{0,n}|$$
$$\leqslant \lim_{\xi\to 0}\varlimsup_{n\to\infty}\left|\frac{1}{\pi}\int_{\frac{\lambda}{n}}^\xi n\Psi(t)\left[\frac{k\gamma'_k(nt)}{t} - \frac{l\gamma'_l(nt)}{t}\right]dt\right.$$
$$\left.-\frac{1}{\pi}\int_{\frac{\lambda}{n}}^\xi \Psi(t)\left[\frac{k\gamma_k(nt)}{t^2} - \frac{l\gamma_l(nt)}{t^2}\right]dt\right|. \tag{38}$$

Now

$$\left|\int_{\frac{\lambda}{n}}^{\xi}\frac{\Psi(t)}{t}n\gamma'_k(nt)dt\right|\leqslant\frac{A}{n^{\beta-1}}\int_{\frac{\lambda}{n}}^{\xi}\frac{|\Psi(t)/t|}{t^{\beta}}dt$$

$$\leqslant\frac{A}{n^{\beta-1}}\cdot\frac{\bar{\theta}(\xi)}{\xi^{\beta-1}}+\frac{A\beta}{n^{\beta-1}}\int_{\frac{\lambda}{n}}^{\xi}\frac{\bar{\theta}(t)}{t^{\beta}}dt,$$

where $\beta=\min(k,3)>1$, and so the expression tends to 0 as $n\to\infty$, $\xi\to 0$ in virtue of the hypothesis on $\bar{\theta}(t)$ if we make $\lambda\to\infty$ after taking $\overline{\lim\limits_{n\to\infty}}$. A similar result holds for the other integrals in (37). Hence

$$\lim_{n\to\infty}\left[S^{(k)}_{0,n}-S^{(l)}_{0,n}\right]=0.$$

This result proves the theorem since the series is summable for $k>2$, and k and l are subject to the sole restriction of being greater than one. The result stated in Theorem 4 is more general than the one obtained by B. N. Prasad. [8]

REFERENCES.

(1) Miss Grimshaw: *Proc. Camb. Phil. Soc.* 23 (1927), 871-81.

(2) G. H. Hardy: *Messenger of Mathematics,* 51 (1922), 185-92.

(3) Hardy and Littlewood; *Proc. L. M. S.* 24 (1925), 211-46.

(4) „ „ : *Journal L. M. S.* 6 (1931), 278-81.

(5) Hobson: *Theory of Functions of a real variable,* Vol. 2 (1926).

(6) Paley: *Proc. Camb. Phil. Soc.* 2 (1930), 173-203.

(7) S. Pollard: „ „ 23 (1926), 373-82.

(8) B. N. Prasad, *Journal de Mathematique,* Tome 11, Fasc. 2, (1932).

SOME INFINITE SERIES INVOLVING SONINE'S POLYNOMIAL

By R. S. VARMA, *Lucknow University.*

[Received 31 July 1939]

1. Sonine's polynomial $T_m^n(x)$ may be defined by the relation

$$T_m^n(x) = \frac{(-)^n}{\Gamma(m+1)\,\Gamma(n+1)}\,{}_1F_1(-n\,;m+1\,;x). \qquad (1)$$

In general we can have two types of infinite series involving the polynomial:

$$\sum_{r=1}^{\infty} a_r T_m^r(x) \qquad (A)$$

and

$$\sum_{r=1}^{\infty} b_r T_r^n(x). \qquad (B)$$

Series of the type (A) have been discussed extensively from time to time by various authors, but very little is known about series of the type (B). The author is only aware of the two infinite series discussed by Erdelyi*, viz.

$$\sum_{r=0}^{\infty} \frac{(-x)^r}{r!} N_{k+\frac{r}{2},\, m+\frac{r}{2}}(t)$$

$$= x^{\frac{1}{2}-\nu} \int_0^t N_{k-\nu,\, m-\nu}(t-y) J_{2\nu-1}(2\sqrt{xy})\, e^{-\frac{1}{2}y} y^{\nu-\frac{1}{2}}\, dy \qquad (2)$$

and

$$\sum_{r=0}^{\infty} \frac{\Gamma(2\nu+r)}{r!} x^r N_{k+\frac{r}{2},\, m+\frac{r}{2}}(t) = \int_0^t y^{2\nu-1} e^{(x-\frac{1}{2})y} N_{k-\nu,\, m-\nu}(t-y)\, dy, \qquad (3)$$

* A. Erdelyi: Funktionalrelationen mit konfluenten hypergeometrischen Funktionen (Zweite Mitteilung), *Math. Zeit.*, 42 (1937), 641-70.

where

$$N_{k,m}(x)=\frac{x^{2m}e^{-\frac{1}{2}x}}{\Gamma(2m+1)}\,{}_1F_1(m+\tfrac{1}{2}-k\,;2m+1\,;x).$$

Since

$$T_m^n(x)=\frac{(-)^n x^{-m}e^{\frac{1}{2}x}}{\Gamma(n+1)}N_{\frac{1}{2}m+n+\frac{1}{2},\frac{1}{2}m}(x);$$

the known series (2) and (3) give for $m=0$ and $k=n+\frac{1}{2}$

$$\sum_{r=0}^{\infty}\frac{(-x)^r}{r!}T_r^n(t)$$

$$=\frac{(-)^n x^{\frac{1}{2}-\nu}e^{\frac{1}{2}x}}{\Gamma(n+1)}\int_0^t N_{n+\frac{1}{2}-\nu,\,-\nu}(t-y)J_{2\nu-1}(2\sqrt{xy})e^{-\frac{1}{2}y}y^{\nu-\frac{1}{2}}\,dy \quad (4)$$

and

$$\sum_{r=0}^{\infty}\frac{\Gamma(2\nu+r)}{r!}x^rT_r^n(t)=\frac{(-)^n e^{\frac{1}{2}x}}{\Gamma(n+r)}\int_0^t y^{2\nu-1}e^{(x-\frac{1}{2})y}N_{n+\frac{1}{2}-\nu,\,-\nu}(t-y)\,dy \quad (5)$$

respectively. If we use the properties of the function $N_{k,m}(x)$, we can show that (4) and (5) further reduce to

$$\sum_{r=0}^{\infty}\frac{(-x)^r}{r!}T_r^n(x)=\frac{1}{n!}\sum_{r=0}^{n}(-)^r\binom{n}{r}x^{\frac{1}{2}(n-r)}J_{n-r}(2\sqrt{x}) \quad (6)$$

and

$$\sum_{r=0}^{\infty}x^rT_r^n(x)=-\frac{1}{n}T_0^{n-1}(x). \quad (7)$$

2. We shall now deduce the nature of b_r in order that a series of the type (B) may converge. The relation (1) shows that, for large m,

$$T_r^n(x)=\frac{(-)^n}{\Gamma(r+1)}\left[1+O(r^{-1})\right]$$

$$=O(e^r r^{-r-\frac{1}{2}}),$$

where we have used Stirling's formula

$$\Gamma(r)=(2\pi)^{\frac{1}{2}}r^{r-\frac{1}{2}}e^{-r}e^{\theta/12r}$$

It follows that if

$$b_r=O(e^{-r}r^{r-\frac{1}{2}-\varepsilon}), \quad (8)$$

where ε is any positive quantity, series of the type (B) will always converge.

In case the series is of the form

$$\sum_{r=1}^{\infty} b_r x^r T_r^n(x), \tag{B'}$$

the condition for convergence is given by the relation

$$\overline{\lim_{r\to\infty}} \sqrt[r]{\left| b_r x^r e^r r^{-r-\frac{1}{2}} \right|} < 1. \tag{9}$$

3. We shall now investigate a few more infiinite series of the type (B′)

From the operational relation*

$$x^m T_m^n(x) \doteqdot \frac{(1-p)^n}{n!\, p^{m+n}}, \tag{10}$$

it follows that

$$\sum_{r=0}^{\infty} \frac{(-)^r}{(2r)!} x^{2r} T_{2r}^n(x) \doteqdot \frac{1}{n!}\left(\frac{1-p}{p}\right)^n \sum_{r=0}^{\infty} \frac{(-)^r}{(2r)!\, p^{2r}}$$

$$= \frac{1}{n!} \frac{1}{p} \cdot \frac{(1-p)^n}{p^{n-1}} \cos(1/p). \tag{11}$$

Interpreting the right-side of (11) by the help of the known result†

$$\text{ber}(2\sqrt{x}) \doteqdot \cos(1/p),$$

we get

$$\sum_{r=0}^{\infty} \frac{(-)^r}{(2r)!} x^{2r} T_{2r}^n(x) = \int_0^x t^{-1} T_{-1}^n(t)\, \text{ber}(2\sqrt{x-t})\, dt$$

$$= \int_0^x (x-t)^{-1} \text{ber}(2\sqrt{t}) T_{-1}^n(x-t)\, dt,\ n > 0, \tag{12}$$

where we have used a known theorem of operational calculus.

* R. S. VARMA: "A pair of functions which are Fourier Sine-Transforms of each other", *Quarterly Jour. of Math.* (Oxford Series), 8 (1938), 203-5.

† Van der Pol: On the operational solution of linear differential equations and an investigation of the properties of these solutions. *Phil. Mag.* (Series 7), 8 (1929), 861-98.

Following the above method and using the results

$$\mathrm{bei}(2\sqrt{x}) \doteqdot \sin(1/p)^*$$

and

$$\sinh x\sqrt{i} - \sin x\sqrt{i} \doteqdot 2i\sqrt{i}\frac{p}{p^4+1}{}^\dagger,$$

we can easily show that

$$\sum_{r=0}^{\infty} \frac{(-)^r}{(2r+1)!} x^{2r+1} T^n_{2r+1}(x)$$

$$= \int_0^x (x-t)^{-1} \mathrm{bei}\,(2\sqrt{t}) T^n_{-1}(x-t)\,dt$$

$$= \int_0^x t^{-1} T^n_{-1}(t)\ \mathrm{bei}(2\sqrt{x-t})\,dt,\ n > 0 \qquad (13)$$

and

$$\sum_{r=0}^{\infty} (-)^r x^{4m} T^n_{4m}(x)$$

$$= \frac{1}{2i\sqrt{i}} \int_0^x t^{-4} T^n_{-4}(t) \left\{ \sinh(x-t)\sqrt{i} - \sin(x-t)\sqrt{i} \right\} dt$$

$$= \frac{1}{2i\sqrt{i}} \int_0^x (x-t)^{-4} T^n_{-4}(x-t) \left\{ \sinh t\sqrt{i} - \sin t\sqrt{i} \right\} dt,$$

$$n > 3. \qquad (14)$$

The series (6) and (7) can also be deduced in the above manner directly from the relation (10).

It is easy to see that the series (12), (13) and (14) investigated above converge by virtue of the condition (9).

* Van der Pol: *loc. cit.*

† P. Humbert: "Some new operational representations", *Proc. Edin. Math. Soc.* (2), 4 (1936), 232-7.